STUDY MATERIAL

SOLUTION SET

SUBJECT: MATHEMATICS

CLASS :XII

KENDRIYA VIDYALAYA SANGATHAN

REGIONAL OFFICE CHANDIGARH

YEAR 2011-2012

TOPIC 1
RELATIONS & FUNCTIONS

(i) Domain , Co domain & Range of a relation

LEVEL I

1. Domain :{ 3, 4, 5}; Range ={3, 4, 5} ; Co-domain={1, 2, 3, 4, 5}
2. $R = \{(1,8), (2,9), (3,10)\}$ Domain $= \{1,2,3\}$ Range $= \{8.9.10\}$

2. Types of relations

LEVEL II

1. Sol: Not reflexive $\because a \neq a - 2 \;\forall\; a \in N \;\&\; a > 6$

2. (i): Reflexive: R is reflexive $\because$ a is divisible by a $\forall$ a $\in$ N

(ii): Symmetry: R is not symmetric $\because$ a is divisible by b

$\&\; b \text{ is not divisble by } a \;\forall a, b \in N$

e.g if $a = 4 \;\&\; b = 2$

4 divisible by 2 but 2 is not divisible by 4

(iii) : Transitive : R is not transitive

Let aRb & bRc $\Rightarrow$ a is divisible by b & *b is divisible by c*

$\Rightarrow a = mb$ for some $m \in Z$ … … … … . (i)

$\Rightarrow b = nc$ for some $n \in Z$ … … … … … . . (ii)

From (i) & (ii), we get $a = mb = m(nc) = (mn)c$ for some $mn \in Z$

$\therefore$ a is a multiple of c $\Rightarrow$ a is divisible by c

$\therefore$ R is reflexive & *Transitive but not symmetric*

3. $R = \{(a, b): a > b\}$

(i) Reflexivity: R is not reflexive $\because a \ngtr a \;\forall\; a \in N$

(ii) Symmetricity: R is not symmetric

Let $(a, b) \in R \;\forall\; a, b \in N$

$\Rightarrow a R b \Rightarrow a > b \Rightarrow b < a \Rightarrow b \ngtr a \Rightarrow (b, a) \notin R$

(iii) Transitivity: R is transitive

Let $(a, b) \in R \;\&(b, c) \in R$

$\Rightarrow a > b$ … … … … … (i)& $b > c$ … … . (ii)

From (i) & $(ii) a > b > c \Rightarrow a > c \Rightarrow (a, c) \in R$

$\therefore$ R is neither reflexive nor symmetric but transitive.

4. $R = \{(a, b) \in R \quad \text{iff} \quad 1 + ab > 0 \forall a, b \epsilon R\}$

(i) Symmetricity:

$$\text{Let}(a, b) \in R \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$$

$\therefore$ R is symmetric

(ii)Reflexivity:

$$R \text{ is reflexive } \because 1 + a.a > 0 \forall a \in R$$

$$\Rightarrow a R a \ \forall a \in R$$

$$(iii) \ Transitivity: Let \ a R b \ \& \ bRc$$

$$\Rightarrow 1 + ab > 0 \ \ \& \ 1 + bc > 0$$

$$Let \ us \ take \ a = 1, b = \frac{-1}{2} \quad , c = -2$$

$$1 + ab = 1 + 1\left(\frac{-1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2} > 0$$

$$\& \ 1 + bc = 1 + \left(\frac{-1}{2}\right)(-2) = 1 + 1 = 2 > 0$$

But $1 + ac = 1 + 1 + (-2) = 1 - 2 = -1 \ngtr 0$

$\therefore$ R is not transitive

5. $: R = \{(x, y) \ / x - 3y = 0 \ i.e \ x = 3y\}$

$$\therefore R = \{(3,1), (6,2), (9,3), (12,4)\}$$

$\therefore$ R is not reflexive $\because (x, x) \notin R \ \forall \ x \in A$

$\therefore$ R is not symmetric $\because (x, y) \in R \ \not\Rightarrow (y, x) \in R \ \forall \ x \ \in A$

$\therefore$ Risnottransitive $\because (x, y) \in R \ \not\Rightarrow (x, z) \in R \ \forall \ x, y, z \ \in A$

LEVEL III

1. $R = \{(a, b): |a - b|$ is multipleof 3

(i) Reflexive : $\because |a - a| = 0$ is multiple of 3

$\therefore$ Relation is reflexive

(ii) Symmetric : Let $(a, b) \in R \Longrightarrow |a-b|$ is divisble by 3

$\Longrightarrow |-(b-a)|$ is divisible by 3 $\Longrightarrow (b, a) \in R$

(ii) Transitive $(a, b) \in R, \quad (b, c) \in R$

$\Longrightarrow |a-b|$ is divisble by 3 & $|b-c|$ is divisble by 3

$\Longrightarrow |(a-b)+(b-c)|$ is divisble by 3

$\Longrightarrow |a-c|$ is divisble by 3 $\Longrightarrow (a, c) \in R$

2. (i) Reflexive $(a, b) R (a, b)$

$\Longrightarrow a+b=b+a$ which is true

$\therefore$ relation is reflexive

(ii)Transitive If $(a, b)R(c, d)$ & $(c, d)R(e, f)$ Then $(a, b) R (e, f)$

Now $(a, b)R(c, d) \Longrightarrow a+d = b+c \quad (1)$

& $(c, d)R(e, f) \Longrightarrow c+f = d+e \quad (2)$

Adding (1)& (2) $a+d+c+f=b+c+d+e$ which is true

$\therefore$ relation is transitive

(iii)Symmetric Now $(a, b)R(c, d) \Longrightarrow a+d = b+c$

$\Longrightarrow d+a=c+b \Longrightarrow (c, d)R(a, b)$

$\therefore$ Relation is symmetric

3.Let A be the set of all the polygons

$R=\{(P_1, P_2): p_1, P_2$ have same no of sides

Reflexive : Since P_1&P_2 have same no of sides

$(P_1, P_2) \in R \; \forall P \in A \therefore R$ is reflexive

Symmetric: Let $(P_1, P_2) \in R$

$\Rightarrow P_1, P_2$ have same no of sides

$\Rightarrow P_2, P_1$ have same no of sides

$\therefore$ R is symmetric

Transitive: Let$(P_1, P_2) \in R$ & $Let(P_2, P_3) \in R$

P_1, P_2 have same no of sides

P_2, P_3 have same no of sides

P_1, P_3 have same no of sides

$\Longrightarrow (P_1, P_2) \in R$

$\therefore$ Ristransitive

Hence R is an equivalence relation

Let B be the set of all the polygons related to the right angled triangle T with sides 3, 4, 5.Therefore set B contains all polygons with three sides.

$\therefore$ B = { P: (P, T) $\in$ R where P is a polygon with three sides.

4. Sol: RonA $= \{x: x \in Z, o \leq x \leq 12\}$

$R = \{(a, b): |a - b| \text{ is multiple of } 3$

Reflexive$|a - a| = 0$ is multiple of 3

$\therefore$ Relation is reflexive

Symmetrica, $b \in R$

$|a - b|$is divisble by 3

$|-(b - a)|$is divisible by 3

$(b, a) \in R$

Transitive$(a, b) \in R, \quad (b, c) \in R$

$|a - b|$is divisble by 3

$|b - c|$ is divisble by 3

$|(a - b) + (b - c)|$ is divisble by 3

$|a - c|$ isdivisble by 3

$(a, c) \in R$

5. Sol: Reflexive$(a, b)R(a, b)$
$a + b = b + a$ which is true

$\therefore$ relation is reflexive

Transitive

$$\text{If}(a, b)R(c, d)$$

$$(c, d)R\,(e, f)$$

Then$(a, b)R(e, f)$

$(a, b)R(c, d)a + d = b + c$

$$(c, d)R(e, f)c + f = d + e$$

Adding $a + d + c + f = b + c + d + e$ which is true

$\therefore$ transitive

$\therefore$ relationistransitive

Symmetric

$$(a, b)R(c, d)$$
$$a + d = b + c$$

$$d + a = c + b$$

$\Rightarrow (c, d)R(a, b)$

$\therefore$ Relationissymmetric

6. A is a set of triangles in a plane

R is a relation defined by

$$R = \{(T_1, T_2): T_1, T_2 \in A \& T_1 \sim T_2\}$$

T. P. T: R is reflexive:

$\forall\ T \in A, T \sim T$ because each triangle is similar to itself

$\therefore$ R is reflexive relation I

T. P. T: R is symmetric:

Let $T_1 R\, T_2 \Rightarrow T_1 \sim T_2$ where $T_1 \& T_2 \in A$

$$\Rightarrow T_2 \sim T_1$$

$$\Rightarrow T_2 R T_1$$

Thus $T_1 R T_2 \Longrightarrow T_2 R T_1$

$\therefore$ R is symmetric relation II

T. P. T: R is transitive

$$\text{Let } T_1, T_2 \& T_3 \in A$$

$$\text{Where } T_1 R T_2 \ \& \ T_2 R\, T_3$$

$\Rightarrow T_1 \sim T_2 \& T_2 \sim T_3$

$\Rightarrow T_1 \sim T_3 \quad \Rightarrow T_1 R T_3$

$\therefore T_1 R T_2 \ \& \ T_2 R T_3 \Rightarrow T_1 R T_3$

$\therefore$ R is transitive III

From I , II & III, R *is equivalence relation*.

Triangle T1 with sides 3,4,5 is similar to triangle T3 with sides 6,8,10

$$\therefore T_1 \sim T_3 \quad \Rightarrow T_1 R T_3$$

(iii)One-one , onto & inverse of a function

LEVEL I

1.Sol: If $f(x) = x^2 - x^{-2} = x^2 - \frac{1}{x^2}$

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 - x^2 = x^{-2} - x^2 = -(x^2 - x^{-2}) = -f(x)$$

$$\therefore f\left(\frac{1}{x}\right) = -f(x)$$

2. Sol: $f: R \rightarrow R$ given $f(x) = x^2$

$$\text{Injective Let } x_1 = 2, x_2 = -2$$

$$f(x_1) = 4$$

$$f(x_2) = 4 \Longrightarrow f(x_1) = f(x_2)$$

Surjective : Let $y = -5 \in R$ be any element

$$f(x) = -5 \Rightarrow x^2 = -5 \Longrightarrow x = \pm\sqrt{-5} \notin R$$

$$\text{for } y = -5 \text{ there is no element in } R \therefore f: R \rightarrow R \text{ is not onto}$$

3. Sol: Injective $f(x) = 2x$

$$\text{Let } x_1, x_2 \in N$$

$$f(x_1) = f(x_2)$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

Surjective : for any $y \in N$ $f(x) = y$

$$\Rightarrow 2x = y$$

$$\Rightarrow x = \frac{y}{2} \quad \forall y \in N \nexists \frac{y}{2} \in N$$

$$f\left(\frac{y}{2}\right) = y \Rightarrow f \text{ is not onto}$$

4. Sol: $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

$$\text{Let } x_1 = 1, x_2 = 3 \text{ be two real numbers}$$

$$f(x_1) = f(1) = 1$$

$$f(x_2) = f(3) = 1$$

$$f(x_1) = f(x_2) \text{ for } x_1 \neq x_2$$

$\therefore$ f: R → R is not one − one

Also $Let\ y = 2 \in R$

$$\text{i.e.} \qquad f(x) = 2$$

$$\text{But Range is } (-1,0,1)$$

$$\text{Clearly there is no value of x for which } f(x) = 2$$

$$\therefore f: R \to R \text{ is not on to}$$

$$\text{Hence the function is neither one} - \text{one nor onto}$$

5. Sol: $A = \{-1,0,1\}$ $B = \{0,1\}$

$$f(x) = x^2$$

$f(-1)\ ^2 = 1$
$f(1)\ ^2 = 1$

It is not one – one. $\therefore$ It is not bijective

6. Sol: $f(x) = \frac{x-1}{x+1}; x \neq 1 \quad f^{-1}(x) = ?$

$$Let y = \frac{x-1}{x+1} \Rightarrow yx + y = x - 1 \Rightarrow yx - x = -1 - y$$

$$\Rightarrow x(y-1) = -(y+1) \Rightarrow x = -\left(\frac{y+1}{y-1}\right)$$

$$\therefore f^{-1}(x) = -\left(\frac{x+1}{x-1}\right)$$

$$f\left(f^{-1}(x)\right) = f\left(-\left(\frac{x+1}{x-1}\right)\right) = f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{2x}{2} = x$$

LEVEL II

1. .yes, f is one-one

2. Sol: Let $y = f(x) \Rightarrow x = f^{-1}(y)$

$$\therefore y = \frac{2x-7}{4}$$

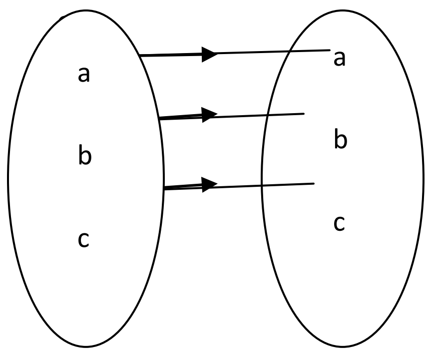
$4y = 2x - 7$

$x = \frac{4y + 7}{2}$

$f^{-1}(y) = \frac{4y + 7}{2}$

$$\Rightarrow f^{-1}(x) = \frac{4x + 7}{2}$$

3. solu: : f: {a, b, c} $\longrightarrow$ {a, b, c}
f *is one* $-$ *onefunctions*



III IV V VI

$Q4$ Sol: f: R $\rightarrow$ Rf(x) $= 7 - 2x^3$

One − one

Let for $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow 7 - 2x_1{}^3 = 7 - 2x_2{}^3$

$\Rightarrow 2x_1{}^3 = 2x_2{}^3$

$\Rightarrow x_1{}^3 = x_2{}^3$

$\Rightarrow x_1 = x_2$

$\therefore f is\ 1 - 1$

Onto Let $f(x) = y$

$$\therefore y = 7 - 2x^3 \Rightarrow 2x^3 = 7 - y \Rightarrow x^3 = \frac{7-y}{2}$$

$$\Rightarrow x = \sqrt[3]{\frac{7-y}{2}}$$

$\therefore$ for every $y \in R \; \exists$ a unique $x \in R$ s.t $f(x) = y$

$\therefore$ f is onto

$\therefore$ f is onto & $1-1$

$Q5$ Sol: $f(x) = \dfrac{3x+5}{2}$

Let $f(x) = y \;\; \therefore y = \dfrac{3x+5}{2} \Rightarrow 2y = 3x+5 \;\; \Longrightarrow x = \dfrac{2y-5}{3}$

$$\Longrightarrow x = \frac{2y-5}{3}$$

Let $g: R \longrightarrow R$ such that $g(y) = \dfrac{2y-5}{3}$

$$gof(x) = g(f(x)) = g\left(\frac{3x+5}{2}\right) = \frac{2\left(\frac{3x+5}{2}\right) - 5}{3} = \frac{3x}{3} = x$$

$\therefore gof(x) = x = I_R$

$$fog(y) = f\big(g(y)\big) = f\left(\frac{2y-5}{3}\right) = \frac{3\left(\frac{2y-5}{3}\right) + 5}{2}$$

$$= \frac{2y-5+5}{2} = \frac{2y}{2} = y$$

$\therefore \; fog(y) = y = I_r$

$\therefore g = f^{-1}$

$\therefore \; f^{-1}(x) = g(x) = \dfrac{2x-5}{3}$

LEVEL III

1. $Sol: \; f: R \rightarrow R \; defined \; by \; f(x) = \frac{2x-1}{3}, x \in R$

T.P.T: $f is\ 1-1$

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$

$$\Rightarrow \frac{2x_1 - 1}{3} = \frac{2x_2 - 1}{3}$$

$$\Rightarrow 2x_1 - 1 = 2x_2 - 1$$

$$\Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow$ fis $1 - 1$

T. P. T: f is onto

Let $f(x) = y$ for $x, y \in R$

$$\Rightarrow \frac{2x-1}{3} = y \text{ or } x = \frac{3y+1}{2}$$

$$\therefore \; \forall \; y \in R, \exists \; x = \frac{3y+1}{2} \in R$$

$\therefore$ f is onto

f is $1-1$, onto & hence invertible

Also $f(x) = y \Rightarrow x = f^{-1}(y) \quad \therefore f^{-1}(y) = \frac{3y+1}{2}$ or $f^{-1}(x) = \frac{3x+1}{2}$

2. Sol: $f: R_+ \rightarrow [-5, \infty)$ defined by $9x^2 + 6x - 5$

(i)**T. P. T f is $1-1$**

Letf$(x_1) = f(x_2)$ for $x_1 = x_2 \in R_+$

$$\Rightarrow 9{x_1}^2 + 6x_1 - 5 = 9{x_2}^2 + 6x_2 - 5$$

$$\Rightarrow 9{x_1}^2 + 6x_1 = 9{x_2}^2 + 6x_2$$

$$\Rightarrow 9({x_1}^2 - {x_2}^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 3(x_1 - x_2) + (3x_1 + 3x_2 + 6) = 0$$

$\Rightarrow 3(x_1 - x_2) = 0 \quad (\because 3x_1 + 3x_2 + 6 \neq 0\ \forall\ x_1, x_2 \in R_+)$

$\Rightarrow x_1 - x_2 = 0 \quad \Rightarrow x_1 = x_2$

Thus , $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2) \Rightarrow \mathrm{x}_1 = \mathrm{x}_2$

$\therefore$ f is $1 - 1$

(ii)**T. P. T**: **fisonto**

Let $\mathrm{y} = 9\mathrm{x}^2 + 6\mathrm{x} - 5$, $\forall\ \mathrm{x} \in \mathrm{R}_+$ where $\mathrm{y} \in$ Range

$\Rightarrow 9\mathrm{x}^2 + 6\mathrm{x} - (5 + \mathrm{y}) = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36 + 36(5 + y)}}{18}$$

$$\Rightarrow x = \frac{6\left[-1 \pm \sqrt{(y + 6)}\right]}{18} \Rightarrow x = \frac{\left[-1 + \sqrt{(y + 6)}\right]}{3} (\because x \in R_+)$$

Thus $\forall$ y $\in$ Range , $\exists$ x $\in \mathrm{R}_+$(Domain) so that f(x) = y

$\Rightarrow$ f is onto

from (i)&(ii), fis $1 - 1$, onto & hence invertible

Also when $\mathrm{f}(\mathrm{x}) = \mathrm{y} \Rightarrow \mathrm{x} = \mathrm{f}^{-1}(\mathrm{y})\ \ \forall\ \mathrm{x} \in \mathrm{R}_+$ & $\mathrm{y} \in [-5, \infty)$

$$\text{so } \mathrm{f}^{-1}(\mathrm{y}) = \frac{\left[-1 + \sqrt{(\mathrm{y} + 6)}\right]}{3}$$

Hence proved

3. Sol: $\mathrm{f}(\mathrm{x}): \mathrm{R} \to \mathrm{R}$, defined by $\mathrm{f}(\mathrm{x}) = 4\mathrm{x} + 3$

(i)T. P. T: f is $1 - 1$

Let $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2)$ for $\mathrm{x}_1, \mathrm{x}_2 \in \mathrm{R}$

$\Rightarrow 4\mathrm{x}_1 + 3 = 4\mathrm{x}_2 + 3$

$\Rightarrow 4(x_1 - x_2) = 0$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$\therefore \mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2) \Rightarrow \mathrm{x}_1 = \mathrm{x}_2$ $\quad\therefore$ f is $1 - 1$

(ii) T.P.T: f is onto

Let $f(x) = y$ for $x \in R$, $y \in R$

$\Rightarrow 4x + 3 = y$

$\Longrightarrow x = \frac{y-3}{4}$ $\forall\, y \in R\ \exists\, x \in R$ such that $x = \frac{y-3}{4}$

$\therefore$ f is onto

f(x) is both 1 − 1, onto. $\therefore$ f(x) is invertible

Also $f(x) = y$ $\forall\, x, y \in R$

$\Longrightarrow x = f^{-1}(y) = \frac{y-3}{4}$

$\therefore f^{-1}(y) = \frac{y-3}{4}$

4. Sol: $f(x): R \rightarrow R$ defined by $f(x) = x^3 + 4$

(i) T.P.T: f is 1 − 1

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$

$\Rightarrow {x_1}^3 + 4 = {x_2}^3 + 4$

$\Rightarrow {x_1}^3 = {x_2}^3 \Rightarrow x_1 = x_2$

Thus, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ for $x_1, x_2 \in R$

$\therefore$ f is 1 − 1

(ii) T.P.T: f is onto

Let $f(x) = y$ for $x, y \in R$

$\Rightarrow x^3 + 4 = y$

$\Rightarrow x^3 = y - 4 \Rightarrow x = (y-4)^{\frac{1}{3}}$

$\forall\, y \in R, \quad \exists\, x \in R \quad s.t\ f(x) = y\ \&\ x = (y-4)^{\frac{1}{3}}$

$\therefore f(x) = y \Rightarrow f^{-1}(y) = x$

$\therefore\ f^{-1}(y) = (y-4)^{\frac{1}{3}}$

(iv) Composition of functions

LEVEL I

1. Sol: $f(x) = e^{2x} g(x) = \log\sqrt{x}, x > 0$

(i) $(f+g)(x) = f(x) + g(x) = e^{2x} + \log\sqrt{x}, x > 0$

(ii) $fg(x) = f(x).\, g(x) = e^{2x}.\log\sqrt{x}, x > 0$

(iii) $fog(x) = f\big(g(x)\big) = f\big(\log\sqrt{x}\big) = e^{2\log\sqrt{x}} = e^{\log(\sqrt{x})^2} = e^{\log x} = x$

(iv) $gof(x) = g(f(x) = g(e^{2x}) = \log\sqrt{e^{2x}} = \log(e^{2x})^{\frac{1}{2}} = \frac{1}{2}\log(e^{2x}) = \frac{2x}{2} = x$

2.Sol: $f(x) = \frac{x-1}{x+1}$

a) $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{1-x}{1+x} = -\frac{(x-1)}{x+1} = -f(x)$

$\therefore f\left(\frac{1}{x}\right) = -f(x)$

b) $f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x}-1}{-\frac{1}{x}+1} = -\frac{(1+x)}{x-1} = -\frac{1}{\frac{x-1}{x+1}} = -\frac{1}{f(x)}$

$\therefore f\left(-\frac{1}{x}\right) = -\frac{1}{f(x)}$

LEVEL II

1. Sol: $f, g: R \to R\, f(x) = |x| \,\&\, g(x) = [x]$

$fog\left(\frac{5}{2}\right) = f\left(g\left(\frac{5}{2}\right)\right) = f\left(\left[\frac{5}{2}\right]\right) = f(2.5]) = f(2) = |2| = 2$

$\therefore\ fog\left(\frac{5}{2}\right) = 2$

$gof(-\sqrt{2}) = g\left(f(-\sqrt{2})\right) = g(|-\sqrt{2}|) = g(\sqrt{2}) = [\sqrt{2}] = [1.414] = 1$

$\therefore gof(-\sqrt{2}) = 1$

2. Sol: $f(x) = \frac{x-1}{x+1}$

$$f(f(x) = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = \frac{x-1-x-1}{x-1+x+1} = -\frac{1}{x}$$

$\therefore f(f(x)) = -\frac{1}{x}$

3. Sol: If $y = f(x) = \frac{3x+4}{5x-3}$

$$fof(x) = f(f(x)) = f\left(\frac{3x+4}{5x-3}\right)$$

$$= \frac{3\left(\frac{3x+4}{5x-3}\right) + 4}{5\left(\frac{3x+4}{5x-3}\right) - 3} = \frac{9x + 12 + 20x - 12}{15x + 20 - 15x + 9} = \frac{29x}{29} = x$$

$\therefore fof(x) = x$

4. As $g \circ f(x) = f \circ g(x) \Longrightarrow g(x) = f^{-1}(x) \Longrightarrow g(x) = \frac{x-7}{10}$

5. As $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$, $f \circ f(x) = f(f(x)) = f\left(\left(3 - x^3\right)^{\frac{1}{3}}\right) = x.$

6. $f(x) = x^2 g(x) = 2x - 3$

$fog(x) = f(g(x)) = f(2x - 3) = (2x - 3)^2 = 4x^2 - 12x + 9$

(v)Binary Operations

LEVEL I

1. Sol: $a * b = L.C.M$ of a, b

$3 * 5 = L.C.M$ of $3,5 = 15$

2. Sol: $a * b = H.C.F$ of a, b

$20 * 16 = H.C.F$ of $20, 16 = 4$

3. Sol: Let e be identity element $a * e = \frac{ae}{5} = a \Longrightarrow e = 5$

4. Sol: $a * b = a + 3b^2$
$2 * 4 = 2 + 3.4^2 = 50$

LEVEL II

1. Sol: $A = NXN$ $(a, b) * (c, d) = (a + c, b + d)$

a)Commutativity

$(a, b) * (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d) * (a, b)$

$\therefore (a, b) * (c, d) = (c, d) * (a, b) \quad \forall\, a, b, c, d \in N$

b) Associativity

Let$(a, b); (c, d); (e, f) \in NXN$

$(a, b) * \big((c, d) * (e, f)\big) = (a, b) * (c + e, d + f)$

$= (a + c + e, b + d + f)$

$\big((a, b) * (c, d)\big) * (e, f) = (a + c, b + d) * (e, f)$

$= (a + c + e, b + d + f)$

$\therefore (a, b) * \big((c, d) * (e, f)\big) = \big((a, b) * (c, d)\big) * (e, f)$

$\therefore$ Associative

(c) Identity : Let (g, h) be the identity element

$\therefore (a, b) * (g, h) = (a, b)$

$(a + g, b + h) = (a, b)$

$\Rightarrow a + g = ab + h = b$

$\therefore g = 0 \quad , \quad h = 0$

But $(0,0) \notin N\ X\ N$

$\therefore$ Identity does not exist

2. Sol: (i) Let $(c, d) \in QXQ$ be an identity element of $*$
where $(a, b) * (c, d) = (ac, ad + b)$.

then $(a, b) * (c, d) = (c, d) * (a, b) = (a, b) \forall (a, b) \in QXQ$

$\therefore (ac, ad + b) = (ca, cb + d) = (a, b)$

$\Rightarrow ac = a = ca \Longrightarrow c = 1$

& $ad + b = cb + d = b \Longrightarrow d = 0 (\because c = 1)$

$\therefore (c, d) = (1,0)$ is the identity of $*$

(ii) Let(e, f) be the invertible element of A

Then$(a, b) * (e, f) = (e, f) * (a, b) = (1,0)$

$\therefore (ae, af + b) = (ea, eb + f) = (1,0)$(Identity element)

$$\Rightarrow \left\{\begin{matrix} ae = 1 = ea \Longrightarrow e = \frac{1}{a} \\ af + b = 0 \Longrightarrow f = \frac{-b}{a} \end{matrix}\right\} \Longrightarrow \text{Inverse element of}(a, b)\text{is}\left(\frac{1}{a}, \frac{-b}{a}\right)$$

3. Sol: (i)$a * b = \frac{a+b}{2}, \quad a, b \in N$

Let $a = 1, b = 2, \quad a * b = 1 * 2 = \frac{3}{2} \notin N \qquad \therefore *$ is not binary

(ii)$a * b = \dfrac{a + b}{2}, \quad a, b \in Q$

$\dfrac{a + b}{2}$ also belongs to Q(Sum Rational No's is also a rational no.)

$\therefore a * b$ is a binary operation

$a * b = b * a \left(\because \dfrac{a + b}{2} = \dfrac{b + a}{2} \; \forall a, b \in Q\right) \qquad \therefore *$ is commutative

$$(a * b) * c = \left(\frac{a + b}{2}\right) * c = \frac{\frac{a+b}{2} + c}{2} = \frac{a + b + 2c}{2} \ldots \ldots . \text{I}$$

$$a * (b * c) = a * \left(\frac{b + c}{2}\right) = \frac{a + \frac{b+c}{2}}{2} = \frac{2a + b + c}{2} \ldots \text{II}$$

From I & II $(a * b) * c \neq a * (b * c)$

$\therefore *$ is not associative

LEVEL III

1.Sol: $*$ is abinary operation on N X N

defined by $(a, b) * (c, d) = (ac, bd) \; \forall (a, b), (c, d) \in N X N$

$(i)(2,3) * (4,1) = (2 * 4, 1 * 3) = (8,3)$

$(ii)[(2,3) * (4,1)] * (3,5) = (8,3) * (3,5) = (24, 15)$

$(2,3) * [(4,1) * (3,5)] = (2,3) * (12,5) = (24, 15)$

$\therefore [(2,3) * (4,1)] * (3,5) = (2,3) * [(4,1) * (3,5)]$

(iii) $\forall$ (a, b) , (c, d), (e, f) $\in$ NXN

Then $(a, b) * (c, d) = (ac, bd)$ & $(c, d) * (a, b) = (ca, db)$

But $(ac, bd) = (ca, db)$ ($\because$ Multi. is comm.)

so $(a, b) * (c, d) = (c, d) * (a, b)$

$\therefore *$ is commutative binary operation

Also $[(a, b) * (c, d)] * (e, f) = (ac, bd) * (e, f) = \big((ac)e, f(bd)\big) \ldots . I$

$(a, b) * [(c, d) * (e, f)] = (a, b) * (ce, df) = \big(a(ce), b(df)\big) \ldots \ldots \ldots \ldots . II$

From I & II , weget

$[(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$($\because$ Multi. is associative)

$\therefore *$ is associative binary operation.

3. *Sol*: *Let us construct a tablefor* $*$

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

From the table it is clear that $0 * 0 = 0; 0 * 1 = 1 * 0 = 1; 0 * 2 = 2 * 0 = 2$ *etc* i.e $0 * a = a * 0 = a \ \forall \ a \in$ given set

Hence 0 is the identity element

Also for each a $\neq 0$ in $\{0,1,2,3,4,5\}$

$6 - a \in \{0,1,2,3,4,5\}$ & $a * (6 - a) = a + (6 - a) - 6 = 0$

Hence $(6 - a)$ is the inverse of a for each $a \neq 0$ in the given set

Note: − Also $0 * 0 = 0$

$\therefore$ 0 is the inverse of itself

where as $6 - 0 \notin \{0,1,2,3,4,5\}$

TOPIC 2
INVERSE TRIGONOMETRIC FUNCTIONS

(i). Principal value branch Table

LEVEL I

1(i)Sol: Let $\cos^{-1}(\frac{\sqrt{3}}{2})$ =y where y $\epsilon[0,\pi]$

⇨$\cos y = \frac{\sqrt{3}}{2}$

⇨$\cos y = \cos\frac{\pi}{6}$

⇨ $y = \frac{\pi}{6}$

Required principal value = $\frac{\pi}{6}$ [$\because \frac{\pi}{6} \in [0, \pi]$

(ii)Sol : Let $\sin^{-1}(-\frac{1}{2})$=y where yϵ $[-\frac{\pi}{2}, \frac{\pi}{2}]$

⇨$\sin y = -\frac{1}{2}$

⇨$\sin y = -\sin\frac{\pi}{6}$

⇨$\sin y = \sin(-\frac{\pi}{6})$

⇨ $y = (-\frac{\pi}{6})$ $\{\because -\frac{\pi}{6} \epsilon [-\frac{\pi}{2}, \frac{\pi}{2}]\}$

(iii)Sol: Let $\tan^{-1}(-\sqrt{3})$ =y where yϵ $[-\frac{\pi}{2}, \frac{\pi}{2}]$

⇨$\tan y = -\sqrt{3}$

⇨$\tan y = -\tan\frac{\pi}{3}$

⇨$\tan y = \tan(-\frac{\pi}{3})$

⇨ $y = -\frac{\pi}{3}$

∴ Required Principal value $= -\frac{\pi}{3}$ [$\because -\frac{\pi}{3} \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv)Sol: Let $\cos^{-1}(-\frac{1}{\sqrt{2}})$ =y, y$\epsilon[0, \pi]$

⇨ $\cos y = -\frac{1}{\sqrt{2}}$

⇨ $\cos y = -\cos\frac{\pi}{4}$

⇨ $\cos y = \cos(\pi - \frac{\pi}{4})$

⇨ $\cos y = \cos\frac{3\pi}{4}$

⇨ $y = \frac{3\pi}{4}$ $(\because \frac{3\pi}{4} \epsilon [0, \pi])$

LEVEL II

1. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$= \left(\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin \pi - \frac{\pi}{3}\right)$

$= \left(\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{\pi}{3}\right)$

$= \left(\frac{2\pi}{3}\right) + \left(\frac{\pi}{3}\right) = \pi$

2. $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$

Sol: $\sin^{-1}(\sin\frac{4\pi}{5}) = \sin^{-1}[\sin(\pi - \frac{\pi}{5})]$

⇨ $\sin^{-1}[\sin(\pi - \frac{\pi}{5})]$

⇨ $= \frac{\pi}{5}$ $\left\{\because \frac{\pi}{5} \epsilon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$

3. $\cos^{-1}(\cos\frac{7\pi}{6})$

Sol: $= \cos^{-1}\left[\cos\pi + \frac{\pi}{6}\right]$

$= \cos^{-1}(-\cos\frac{\pi}{6})$

$= \cos^{-1}(-\frac{\sqrt{3}}{2})$

$= \pi - \cos^{-1}\frac{\sqrt{3}}{2}$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

(ii). Properties of Inverse Trigonometric Functions

LEVEL I

1. Sol: $\Rightarrow \cot(\tan^{-1} a + \cot^{-1} a) = \cot\frac{\pi}{2} = 0$

$\left\{\because \tan^{-1} a + \cot^{-1} a = \frac{\pi}{2}\right\}$

2: $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

Sol: R.H.S $= \sin^{-1}(3x - 4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$ $\left\{\text{Put } x = \sin\theta \; \therefore \theta = \sin^{-1} x\right.$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1} x = \text{L.H.S}$

3. Find x if $\sec^{-1}\left(\sqrt{2}\right) + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

Sol: $\frac{\pi}{4} + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

$\operatorname{cosec}^{-1} x = \frac{\pi}{2} - \frac{\pi}{4}$

Or $x = \operatorname{cosec}\frac{\pi}{4}$

$x = \sqrt{2}$

LEVEL II

1. Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

2. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Sol: Let $\sin^{-1}\frac{8}{17} = x \Rightarrow \sin x = \frac{8}{17}$

$= \sin^{-1}\frac{3}{5} = y \Rightarrow \sin y = \frac{3}{5}$

$= \cos x = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{298}}$

$= \sqrt{\frac{225}{289}} = \frac{15}{17}$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}}$

$= \frac{4}{5}$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{8}{15}$

$\therefore \tan y = \frac{\sin y}{\cos y} = \frac{3}{4}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$$=\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}*\frac{3}{4}} = \frac{32+45}{60-24} = \frac{77}{36}$$

$$\therefore x + y = \tan^{-1}\frac{77}{36}$$

$$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$$

3. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

$$L.H.S. = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}$$

$$= tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{8}$$

$$= tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}} + tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}$$

$$= tan^{-1}\frac{\frac{8}{15}}{1-\frac{1}{15}} + tan^{-1}\frac{\frac{15}{56}}{1-\frac{1}{56}}$$

$$= tan^{-1}\frac{\frac{8}{15}}{\frac{14}{15}} + tan^{-1}\frac{\frac{15}{56}}{\frac{55}{56}}$$

$$= tan^{-1}\frac{8}{14} + tan^{-1}\frac{15}{55}$$

$$= tan^{-1}\frac{\frac{8}{14}+\frac{15}{55}}{1-\frac{8}{14}\times\frac{15}{55}}$$

$$= tan^{-1}\frac{650}{650} = tan^{-1} 1 = \frac{\pi}{4}$$

4. $\text{L.H.S.} = 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{\frac{3}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right)$$

LEVEL III

1. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, \; x \in \left(0, \frac{\pi}{4}\right)$

Sol: L.H.S $= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left(\frac{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}-\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)+\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)$$

$$= \cot^{-1}\left[\frac{1+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{1-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$= \cot^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$= \cot^{-1}\left[\cot\frac{x}{2}\right] = \frac{x}{2} = \text{R.H.S}$$

2. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

Sol: $tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Put $x = cos2\theta$

$= tan^{-1}\left[\frac{\sqrt{1+cos2\theta}-\sqrt{1-cos2\theta}}{\sqrt{1+cos2\theta}+\sqrt{1-cos2\theta}}\right]$

$= tan^{-1}\left[\frac{\sqrt{2}cos\,\theta-\sqrt{2}sin\theta}{\sqrt{2}cos\,\theta+\sqrt{2}sin\theta}\right]$

$= tan^{-1}\left[\frac{1-tan\theta}{1+tan\theta}\right] = tan^{-1}\left[tan\left(\frac{\pi}{4}-\theta\right)\right]$

$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x = R.H.S$

3. $\tan^{-1}2x + \tan^{-1}3x = \pi/4$

Sol: $tan^{-1}2x + tan^{-1}3x = \frac{\pi}{4}$

$\Rightarrow tan^{-1}\frac{2x+3x}{1-2x.3x} = \frac{\pi}{4}$

$\Rightarrow tan^{-1}\frac{5x}{1-6x^2} = \frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = tan\frac{\pi}{4}$

$\Rightarrow 5x = 1 - 6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow 6x^2 + 6x - x - 1 = 0$

Here, $x \neq -1$ $\forall$ they don't satisfy the given eq.

$\Rightarrow 6x(x+1) - 1(x+1) = 0$ $\quad\quad$ $\therefore x = \frac{1}{6}$ only answer

$\Rightarrow x = -1, x = \frac{1}{6}$

4. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

Sol: $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

$$\Rightarrow \tan^{-1}\left\{\frac{(x+1)+(x-1)}{1-(x+1).(x-1)}\right\} = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-(x^2+1)}\right) = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{-x^2}\right) = \tan^{-1}\frac{8}{31} \Rightarrow \frac{2x}{-x^2} = \frac{8}{31}$$

$$\Rightarrow \frac{2}{-x} = \frac{8}{31} \Rightarrow 62 = -8x \Rightarrow x = -\frac{31}{4}$$

5. $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

Sol: $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}*\frac{x+1}{x+2}}\right] = \frac{\pi}{4} \quad [\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}]$$

$$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+1)-(x-1)(x+1)} = tan\frac{\pi}{4}$$

$$\Longrightarrow \frac{x^2+x-2+x^2-x-2}{x^2-4-(x^2-1)} = 1$$

$$\Longrightarrow \frac{2x^2-4}{-3} = 1$$

$$\Longrightarrow 2x^2-4 = -3 \Longrightarrow 2x^2 = 1$$

$$\Longrightarrow x^2 = \frac{1}{2}$$

$$\Longrightarrow x = \pm\frac{1}{\sqrt{2}}$$

TOPIC -3
MATRICES & DETERMINANTS

(i). Order, Addition, Multiplication and transpose of matrices:

LEVEL I

1. The possible orders: $5 \times 1, \ 1 \times 5$

2. $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

3. $\begin{bmatrix} -3 & -4 & 1 \\ 1 & 1 & -3 \end{bmatrix}$

4. The order of AB is $2 \times 2$and BA is 3×3

LEVEL II

1. We have A = $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, B = $[-1 \quad 2 \quad 1]$

A^T =$[1 \quad -4 \quad 3]$$B^T$=$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

AisoAB =$\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}[-1 \quad 2 \quad 1]$and $B^T A^T$ = $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}[1 \quad -4 \quad 3]$

= $\begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$ = $\begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$

$(AB)^T$ = $\begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$Hence verified $(AB)^T = B^T A^T$

3. If B is skew symmetric matrix,$\Rightarrow B^T = -B$

Now $(ABA^T)^T = (ABA^T)^T$ as $(AB)^T = B^T A^T$

$= (A^{T)})^T (AB)^T$

$= A(-B)A^T$

$= -ABA^T$

4. Hence the matrix (ABA^T) is skew symmetric.

$$A^2 = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

$$aI = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}a & 0\\0 & a\end{bmatrix}$$

$$bA = b\begin{bmatrix}3 & 1\\7 & 5\end{bmatrix} = \begin{bmatrix}3b & 1b\\7b & 5b\end{bmatrix}$$

As $A^2 + aI = bA \Longrightarrow \begin{bmatrix}16 & 8\\56 & 32\end{bmatrix} + \begin{bmatrix}a & 0\\0 & a\end{bmatrix} = \begin{bmatrix}3b & 1b\\7b & 5b\end{bmatrix}$

$$\Longrightarrow \begin{bmatrix}16+a & 8\\56 & 32+a\end{bmatrix} = \begin{bmatrix}3b & 1b\\7b & 5b\end{bmatrix} \Longrightarrow a = 8 \;\& b = 8$$

LEVEL III

.

1. $A^2 = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix} = \begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix}$

$$3A = 3\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix} = \begin{bmatrix}6 & 0 & 3\\6 & 3 & 9\\3 & -3 & 0\end{bmatrix}$$

$$2I = 2\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$$

$$A^2 - 3A + 2I = \begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix} - \begin{bmatrix}6 & 0 & 3\\6 & 3 & 9\\3 & -3 & 0\end{bmatrix} + \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & -1 & -1\\-3 & -3 & -4\\-3 & 2 & 0\end{bmatrix}$$

2. We have $A = \begin{bmatrix}3 & -2 & -4\\3 & -2 & -5\\-1 & 1 & 2\end{bmatrix} \Longrightarrow A^T = \begin{bmatrix}3 & 3 & -1\\-2 & -2 & 1\\-4 & -5 & 2\end{bmatrix}$

$$A + A^T = \begin{bmatrix}3 & -2 & -4\\3 & -2 & -5\\-1 & 1 & 2\end{bmatrix} + \begin{bmatrix}3 & 3 & -1\\-2 & -2 & 1\\-4 & -5 & 2\end{bmatrix} = \begin{bmatrix}6 & 1 & -5\\1 & -4 & -4\\-5 & -4 & 4\end{bmatrix}$$

$$\frac{1}{2}\left(A + A^T\right) = \frac{1}{2}\begin{bmatrix}6 & 1 & -5\\1 & -4 & -4\\-5 & -4 & 4\end{bmatrix} = \begin{bmatrix}3 & \frac{1}{2} & -\frac{5}{2}\\ \frac{1}{2} & -2 & -2\\ -\frac{5}{2} & -2 & 2\end{bmatrix}$$

$$A - A^T = \begin{bmatrix}3 & -2 & -4\\3 & -2 & -5\\-1 & 1 & 2\end{bmatrix} - \begin{bmatrix}3 & 3 & -1\\-2 & -2 & 1\\-4 & -5 & 2\end{bmatrix} = \begin{bmatrix}0 & -5 & -3\\5 & 0 & -6\\3 & 6 & 0\end{bmatrix}$$

$$\frac{1}{2}\left(A - A^T\right) = \frac{1}{2}\left[\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

$$Hence A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

$$\boldsymbol{A} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

3. Step1: $A^1 = \begin{bmatrix} a^1 & \frac{b(a^1-1)}{a-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

Thus result is true for n = 1

Step 2: Assume the result is true for n = m, where $1 \leq m \leq n$

i.e. $A^m = \begin{bmatrix} a^m & \frac{b(a^m-1)}{a-1} \\ 0 & 1 \end{bmatrix}$

Step 3: $A^{m+1} = A^m A = \begin{bmatrix} a^m & \frac{b(a^m-1)}{a-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

$$\Longrightarrow A^{m+1} = \begin{bmatrix} a^{m+1} & \frac{b(a^{m+1}-1)}{a-1} \\ 0 & 1 \end{bmatrix}$$

i.e. the result is true for n = m+1, whenever it is true for n = m

Hence it is true for every value of n , by principle of mathematical induction.

(ii) Cofactors &Adjoint of a matrix

LEVEL I

1. The co-factor of a_{12} is $(-1)^3[-42 - 4] = 46$
2. C_{11} = Cofactor of A_{11} $= 3$

   C_{12} = Cofactor of A_{12} $= -4$

   C_{21} = Cofactor of A_{21} $= 1$

   C_{22} = Cofactor of A_{22} $= 2$

   AdjA $= \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$

LEVEL II

1. Here $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ & $AdjA = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ & $|A| = 0$

$$A(AdjA) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \&(AdjA)A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A(adjA) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (adjA)A = |A|I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Here$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ & $AdjA = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} |A| = -7$

$$NowA(AdjA) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$Also(AdjA)A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$ThusA(adjA) = \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = (adjA)A = |A|I = (-7)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii)Inverse of a Matrix & Applications

LEVEL I

1. Here $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} \&|A| = -19$

   $AdjA = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

   $A^{-1} = \frac{AdjA}{|A|} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}A$

2. As $A^2 = I \Longrightarrow A^2A^{-1} = IA^{-1} \Longrightarrow AAA^{-1} = IA^{-1} \Longrightarrow AI = A^{-1} \Longrightarrow A = A^{-1}$
3. The matrix $A = \begin{bmatrix} 2-k & 3 \\ -5 & 1 \end{bmatrix}$ is not invertible, if$|A| = 0$

   i.e. $15 + (2-k) = 0 \quad \Longrightarrow \quad k = 17$

LEVEL II

1. $A^2 = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$

$5A = 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix}$

$14\,I = 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$A^2 - 5A - 14\,I = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore, $\mathrm{A}^2 - 5\mathrm{A} - 14\,\mathrm{I} = 0$

(ii)To find A^{-1}: $Now,\quad A^2 - 5A - 14\,I = 0$

$\Longrightarrow \quad A^2 - 5A = 14\,I$

$\Longrightarrow \quad A^2A^{-1} - 5AA^{-1} = 14\,IA^{-1}$

$\Longrightarrow A - 5I = 14\,A^{-1}$

$\Longrightarrow \quad A^{-1} = \frac{1}{14}(A - 5I) \quad = -\frac{1}{14}\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$

LEVEL III

1. Let $B = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 2 & 1 \\ 5 & -4 & -2 \end{bmatrix}, C = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let $\mathrm{BX} = \mathrm{C} \Longrightarrow \mathrm{X} = \mathrm{B}^{-1}\mathrm{C} = (\mathrm{A}^{\mathrm{T}})^{-1}\mathrm{C} = (\mathrm{A}^{-1})^{\mathrm{T}}\mathrm{C}$

As we know, $A^{-1} = \frac{1}{|A|}(AdjA)$

$A = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 2 & 1 \\ 5 & -4 & -2 \end{bmatrix} \Longrightarrow |A| = \begin{vmatrix} 2 & 3 & 1 \\ -3 & 2 & 1 \\ 5 & -4 & -2 \end{vmatrix}$

$= 2(-4 + 4) - 3(6 - 5) + 1(12 - 10)$

$= 0 - 3 + 2 = -1$

$adjA = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$ & $A^{-1} = \frac{1}{|A|}(AdjA) = \frac{1}{-1}\begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$

$(A^{-1})^T = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 9 & 5 \\ -2 & -23 & -13 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

$$\Rightarrow (A^{-1})^T C = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = 1, y = 2, z = 3$$

2. Given. x + 2y - 3z = - 4 , 2x + 3y + 2z = 2, 3x - 3y – 4z = 11

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = -6 + 28 + 45 = 67$$

$$\Rightarrow \ AdjA = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}(AdjA) = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 24+34+143 \\ -56+10-88 \\ 60+18-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \Rightarrow x = 3, y = -2, z = 1$$

3. Given $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$\Rightarrow \text{AB} = \begin{bmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow A\text{B} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I$$

$$|\text{A}| \neq 0 \Rightarrow \text{A is invertible and } \text{A}^{-1} = \frac{1}{6}\text{B} \ \because \text{AB} = 6\text{I} \Rightarrow \text{I} = \frac{1}{6}\text{AB}$$

Now given system, $x - y = 3$, $2x + 3y + 4z = 17$, $y + 2z = 7$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\Rightarrow AX = C \Rightarrow X = A^{-1}C$$

$$\Rightarrow X = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$\Rightarrow x = 2, \qquad y = -1, \qquad z = 4$$

4. Put $\frac{1}{x} = X, \frac{1}{y} = Y, \frac{1}{z} = Z \quad \Rightarrow 2X - 3Y + 3Z = 10 \ldots\ldots(i)$

$$X + Y + Z = 10 \ldots\ldots(ii)$$

$$3X - Y + 2Z = 13 \ldots\ldots(iii)$$

Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}, C = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \Rightarrow AC = B \quad \Rightarrow C = A^{-1}B$

$$\Rightarrow \ A^{-1} = \frac{1}{|A|}\ (AdjA)$$

$$\Rightarrow |A| = 2(2+1) + 3(2-3) + 3(-1-3)$$

$$= 6 - 3 - 12 = -9$$

$$= AdjA = \begin{bmatrix} 3 & 1 & -4 \\ 3 & -5 & -7 \\ -6 & 1 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}$$

$$\Rightarrow \ A^{-1} = \frac{1}{|A|}\ (AdjA) \Rightarrow \frac{1}{-9}\begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}$$

$$\Rightarrow \ C = A^{-1}B \Rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{1}{-9}\begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}$$

$$= \frac{1}{-9}\begin{bmatrix} 30 & 30 & -78 \\ 10 & -50 & 13 \\ -40 & -70 & 65 \end{bmatrix} = \frac{1}{-9}\begin{bmatrix} -18 \\ -27 \\ -45 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$\Rightarrow X = 2, Y = 3, Z = 5 \quad \therefore x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$

5. We know that A = IA $\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A

$$Operate R_2 \longrightarrow R_2 + R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$Operate R_1 \longrightarrow R_1 + R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$Operate R_2 \longrightarrow R_2 + 2R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$Operate R_3 \longrightarrow R_3 + 2R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$Operate R_1 \longrightarrow R_1 + R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

(iv)To Find The Difference Between $|A|, |adjA|, |kA|$

1. $cos\,15^\circ\, cos\,75^\circ - sin\,15^\circ\, sin\,75^\circ = cos(15 + 75) = cos\,90^\circ = 0$
2. $|3I| = 3^3 = 27$
3. $|2A| = 2^3|A| = 8 \times 3 = 24$
4. The given matrix is a singular matrix if

$$\begin{vmatrix} 2a & -1 \\ -8 & 3 \end{vmatrix} = 0 \Rightarrow 6a - 8 = 0 \Rightarrow a = \frac{4}{3}$$

LEVEL II

1 $\text{As}|\text{adj}A| = |A|^{n-1}, \Rightarrow |A|^{n-1} = |(\text{adj}A)| = 64 = 8^{3-1} \Rightarrow |A| = 8$
$\text{Also}|A'| = |A| \Rightarrow |A'| = 8$

2. $\text{As}|\text{adj}A| = |A|^{n-1}, \Rightarrow |(\text{adj}A)| = 7^{3-1} = 49$

LEVEL III

1. $\text{As}A = \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} \Rightarrow |A| = a^2 - 4$

$|A|^3 = 125 \Rightarrow |A| = 5 \Rightarrow a^2 - 4 = 5 \Rightarrow a = \pm 3$

2. $|\text{adj}A| = |A|^{n-1}, |A(\text{adj}A)| = |A||\text{adj}A| = |A||A|^{n-1} = 125$

(v).Properties of Determinants

LEVEL I

1. $\begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}$

$$\Rightarrow 32 - 15 = 2x^2 - 15$$
$$\Rightarrow x^2 = 16 \quad x = 4$$

2. $= (a+ib)(a-ib) - (-c+id)(c+id)$
$= (a+ib)(a-ib) + (c-id)(c+id)$
$= (a^2+b^2) + (c^2+d^2)$
$= (a^2+b^2) + c^2 + d^2$

LEVEL II

1. Sol: $R_1 \to R_1 - (R_2 + R_3)$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Taking -2 common from R_1

$$= -2\begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Operate $R_2 \longrightarrow R_2 - R_1 \& R_3 \longrightarrow R_3 - R_1$

$$= -2\begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$$

Expand with R_1

$$= -2\,(-c(ab) + b(-ac)) = (-2)(-2abc)$$

$$= (4abc) \quad \text{Hence Proved}$$

2. Solu. Operate $C_1 \longrightarrow C_1 - bC_3, C_2 \longrightarrow C_2 + aC_3$

$$\Rightarrow L.H.S = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking $(1 + a^2 + b^2)$ from C_1& same from C_2

$$\Rightarrow (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Operate $R_3 \longrightarrow R_3 - bR_1 + aR_2$

$$\Rightarrow (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

Expand with C_1

$$\Rightarrow (1+a^2+b^2)^2 1.(1+a^2+b^2) = (1+a^2+b^2)^3$$

3. $L.H.S. = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= [1 + pxyz] \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Operate $R_2 \longrightarrow R_2 - R_1$ and $R_3 \longrightarrow R_3 - R_1$

$$= [1 + pxyz] \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

Taking $(y - x)$ from R_2 and $(z - x)$ from R_3

$$= [1 + pxyz](y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

Expanding from C_1

$$= (1 + pxyz)(x - y)(y - z)(z - x)$$

LEVEL III

1. (a) $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Operate $R_1 \longrightarrow R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3a-x & 3a-x & 3a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Taking $(3a - x)$ from R_1

$$\Rightarrow (3a-x) \begin{vmatrix} 1 & 1 & 1 \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Operate $C_2 \longrightarrow C_2 - C_1$ and $C_3 \longrightarrow C_3 - C_1$

$$\Rightarrow (3a-x) \begin{vmatrix} 1 & 0 & 0 \\ a-x & 2x & 0 \\ a-x & 0 & 2x \end{vmatrix} = 0$$

Expanding from R_1

$$\Rightarrow (3a - x)(4x^2) = 0$$

$$\Rightarrow x = 0, 0, 3a$$

b. $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Operate $R_1 \longrightarrow R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Taking $(3x + a)$ from R_1

$$\Rightarrow (3x+a) \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Operate $C_2 \longrightarrow C_2 - C_1$ and $C_3 \longrightarrow C_3 - C_1$

$$\Rightarrow (3x+a) \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding from R_1

$\Rightarrow$ $(3x + a)(a^2) = 0$ $\Rightarrow$ $x = \frac{-a}{3}$

2. $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\Rightarrow$ $\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$ (Operate $R_1 \longrightarrow R_1 + R_2 + R_3$)

$\Rightarrow$ $\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$ (Taking (a + b + c) from R_1)

$\Rightarrow$ $\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$ Operate $C_2 \longrightarrow C_2 - C_1$ and $C_3 \longrightarrow C_3 - C_1$

$\Rightarrow$ $\Delta = (a+b+c)[(c-b)(b-c)-(a-b)(a-c)$ Expanding from R_1

$\Rightarrow$ $\Delta = (a+b+c)[(c-b)(b-c)-(a-b)(a-c)]$

$\Rightarrow$ $\Delta = (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

$\Rightarrow$ $\Delta = -\frac{\mathbf{1}}{\mathbf{2}}(a+b+c)\left((a-b)^2+(b-c)^{2+}(a-c)^2\right)$

This is always negative $\because$ $a + b + c > 0$

3. Solu. L.H.S = Operate $R_1 \longrightarrow \frac{1}{a}R_1, R_2 \longrightarrow \frac{1}{b}R_2, R_3 \longrightarrow \frac{1}{c}R_3$

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

$C_1 \longrightarrow aC_1,\ C_2 \longrightarrow bC_2,\ C_3 \longrightarrow cC_3$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$C_1 \longrightarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} a^2+b^2+c^2+1 & b^2 & c^2 \\ a^2+b^2+c^2+1 & b^2+1 & c^2 \\ a^2+b^2+c^2+1 & b^2 & c^2+1 \end{vmatrix}$$

Taking common $(a^2+b^2+c^2+1)$ from C_1

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

Operate $R_2 \longrightarrow R_2 - R_1, R_3 \longrightarrow R_3 - R_1$

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & c^2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a^2+b^2+c^2+1).1 = (a^2+b^2+c^2+1)$$

4. L.H.S $\quad R_1 \longrightarrow R_1 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

$R_2 \longrightarrow R_2 + R_3$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a+c & b+a & c+b \\ b+c & c+a & a+b \end{vmatrix}$$

$C_2 \longrightarrow C_2 - C_1, C_3 \longrightarrow C_3 - C_1$

$$\Longrightarrow (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ a+c & b-c & b-a \\ b+c & a-b & a-c \end{vmatrix}$$

Expand with C_1

$$= (a+b+c)((b-c)(a-c)-(b-a)(a-b))$$

$$= (a+b+c)(ab-ac-bc+c^2-(ab-a^2-b^2+ab))$$

$= (a + b + c)(ab - ac - bc + c^2 - ab + a^2 + b^2 - ab)$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$

5. Operate $R_1 \longrightarrow aR_1, R_2 \longrightarrow bR_2, R_3 \longrightarrow cR_3$

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & a(b+c) \\ bc^2a^2 & bca & b(c+a) \\ ca^2b^2 & cab & c(a+b) \end{vmatrix}$$

Taking abc common from C_2

$$= \frac{abc}{abc}\begin{vmatrix} ab^2c^2 & 1 & ab+ac \\ bc^2a^2 & 1 & bc+ab \\ ca^2b^2 & 1 & ca+bc \end{vmatrix}$$

Taking abc common from C_1

$$= abc\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ca+bc \end{vmatrix}$$

$C_1 \rightarrow C_1 + C_3$

$$=abc\begin{vmatrix} bc+ac+ab & 1 & ab+ac \\ ba+bab+ca & 1 & bc+ab \\ ab+bc+ca & 1 & ca+bc \end{vmatrix}$$

Taking $(bc + ac + ab)$ common from C_1

$$= (abc)(ab+bc+ca)\begin{vmatrix} 1 & 1 & ab+ac \\ 1 & 1 & bc+ab \\ 1 & 1 & ca+bc \end{vmatrix}$$

Two columns are identical in value of determinant is 0

$\therefore (abc)(ab + bc + ca) * 0 = 0$

6. Multiply $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3,$

$$= \frac{1}{abc}\begin{vmatrix} -bac & b^2a+abc & ac^2+abc \\ a^2b+abc & -bac & bc^2+abc \\ a^2c+abc & cb^2+abc & -abc \end{vmatrix}$$

Taking b, a, c common from C_1, C_2, C_3

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ba+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+ab & bc+ac & -ab \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} ab+bc+ca & bc+ba+ac & ac+ab+bc \\ ab+bc & -ac & bc+ab \\ ac+ab & bc+ac & -ab \end{vmatrix}$$

Taking $(ab+bc+ca)$ as common from R_1

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+ab & bc+ac & -ab \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -ac-ab-bc & 0 \\ ac+ab & bc-ab & -ab-bc-ca \end{vmatrix}$$

Expnding along R_1

$$= (ab+bc+ca)[(ac+ab+bc)(ac+ab+bc)]$$

$$= (ab+bc+ca)^3$$

7. $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

Solu: $L.H.S. = \begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix}$

$$= abc\begin{vmatrix} \frac{(b+c)^2}{a} & a & a \\ b & \frac{(a+c)^2}{b} & b \\ c & c & \frac{(a+b)^2}{c} \end{vmatrix} \text{(Taking a from} C_1\text{, b from } C_2 \text{ \& } c \text{ from} C_3\text{)}$$

$$= \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} \text{(Multiplying } R_1 \text{ by a, } R_2 \text{ by b \& } R_3 \text{by c)}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (a+c)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix} \text{Operate } R_1 \longrightarrow R_1 - R_3 \text{ and } R_2 \longrightarrow R_2 - R_3$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} (\text{Taking } (a+b+c) \text{ from } C_1 \& C_2)$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \text{Operate } R_3 \longrightarrow R_3 - R_1 - R_2$$

$$= (a+b+c)^2 \begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(a+c) & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \text{Operate } C_1 \longrightarrow aC_1 + C_3 \text{ and } C_2 \longrightarrow bC_2 + C_3$$

$$= 2abc(a+b+c)^3 \text{Expanding from } R_3$$

8. If p, q, r are not in G.P and $\begin{vmatrix} 1 & \frac{q}{p} & \alpha + \frac{q}{p} \\ 1 & \frac{r}{q} & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$, show that $p\alpha^2 + 2p\alpha + r = 0$.

Apply $R_1 \to R_1 - R_2$ $\Rightarrow \begin{vmatrix} 0 & \frac{q}{p} - \frac{r}{q} & \frac{q}{p} - \frac{r}{q} \\ 1 & \frac{r}{q} & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$

Taking$(\frac{q}{p} - \frac{r}{q})$ common fromR_1

$$\Rightarrow \left(\frac{q}{p} - \frac{r}{q}\right) \begin{vmatrix} 0 & 1 & 1 \\ 1 & \frac{r}{q} & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$$

Apply $C_2 \to C_2 - C_3$

$$\Rightarrow \left(\frac{q}{p} - \frac{r}{q}\right) \begin{vmatrix} 0 & 0 & 1 \\ 1 & -\alpha & \alpha + \frac{r}{q} \\ p\alpha + q & q\alpha + r & 0 \end{vmatrix} = 0$$

Expnding with R_1

$$\Longrightarrow \frac{q^2 - pr}{pq}\{-p\alpha^2 - q\alpha - q\alpha - r\} = 0$$

$$\Longrightarrow \frac{q^2 - pr}{pq}\{p\alpha^2 + 2q\alpha + r\} = 0$$

$$\because \ \mathrm{q^2 - pr} \neq 0 \text{ as } \mathrm{p, q, r} \text{ are not in GP}$$

$$\Longrightarrow \mathrm{p\alpha^2 + 2q\alpha + r = 0}\text{ . Hence proved}$$

9. L.H.S: Operate $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2b + 2c + 2a & 2c + 2a + 2b & 2a + 2b + 2c \\ c + a & a + b & b + c \\ a + b & b + c & c + a \end{vmatrix}$$

Taking (a+b+c) common from R_1

$$= 2(a + b + c)\begin{vmatrix} 1 & 1 & 1 \\ c + a & a + b & b + c \\ a + b & b + c & c + a \end{vmatrix}$$

$$C_2 \longrightarrow C_2 - C_1, C_3 \longrightarrow C_3 - C_1$$

$$= 2(a + b + c)\begin{vmatrix} 1 & 0 & 0 \\ c + a & b - c & b - a \\ a + b & c - a & c - b \end{vmatrix}$$

Expand with R_1

$$= 2(a + b + c)((b - c)(c - b) - (b - a)(c - a))$$

$$= -2(a + b + c)(bc - b^2 - c^2 + bc - bc + ac + ab - a^2)$$

$$= -(a + b + c)(2bc - 2b^2 - 2c^2 + 2ac + 2ab - 2a^2)$$

$$= (a + b + c)(-2bc + 2b^2 + 2c^2 - 2ac - 2ab + 2a^2)$$

$$= (a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)$$

Given $\Delta = 0 \Longrightarrow$ Either $(a + b + c) = 0$

$$or((a - b)^2 + (b - c)^2 + (c - a)^2) = 0$$

$$\Longrightarrow a = b, b = c, c = a \Longrightarrow a = b = c$$

TOPIC 4
CONTINUITY AND DIFFRENTIABILITY
LEVEL I

2.Continuity

1:- $f(x) = x^2+5$

L.H.L. $\lim_{x\to -1^-} f(x) = \lim_{h\to 0}(-1-h)^2+5$

$= \lim_{h\to 0} 1+h^2+2h+5$

$= \lim_{h\to 0} 6+h^2+2h$

= 6

R.H.L $\lim_{x\to -1^+} f(x) = \lim_{h\to 0}(-1+h)^2+5$

$= \lim_{h\to 0} 1+h^2-2h+5$

$= \lim_{h\to 0} 6+h^2-2h$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 6$

$f(-1) = 6$

L.H.L. = R.H.L. = $f(-1)$

$\therefore$ f(x) is continuous at $x = -1$

2. $f(x) = \frac{1}{x+3}$

Not continuous because x ∈ R

3. $f(x)$= 4x $\forall x \in R$

It is continuous, as 4x is a polynomial function

LEVEL II.

2.. $f(x)$ is continuous at x=2

$\therefore L.H.L = R.H.L. = f(2)$

L.H.L. =kx^2= 4k

R.H.L. = 3

$f(2)$ = 4k

4k= 3

$\therefore k = \frac{3}{4}$

Q3 Sol.

f(x) = $\begin{cases} ax + 1 & \text{if } x \leq 3 \\ bx + 3 & \text{if } x > 3 \end{cases}$ is continuous at x=3.

$\text{At } x = 3$
$\text{LHL} = \lim_{x \to 3} ax + 1 = 3a + 1$

$\text{RHL} = \lim_{x \to 3} bx + 3 = 3b + 3 = 3(b + 1)$
$\text{As f is continous at } x = 3 \Rightarrow a - b = \frac{2}{3}$

4. If 4.f(x) = $\begin{Bmatrix} \frac{\sin 3x}{x}, x \neq 0 \\ 1, \quad x = 0 \end{Bmatrix} \lim_{h \to 0}$

At x=0 L.H.L = $\lim_{h \to 0^-} \frac{\sin 3x}{x}$

$= \lim_{h \to 0} 3\left(\frac{\sin(-3h)}{-3h}\right)$

$= 3 \lim_{h \to 0} \left(\frac{-\sin 3h}{-3h}\right)$

= 3

R.H.L. = $\lim_{h \to 0^+} \frac{\sin 3x}{x} = \lim_{h \to 0} 3\left(\frac{\sin(3h)}{3h}\right) = 3$

f(0) = 1

L.H.L.= R.H.L. $\neq f(0)$

∴ f(x) is not continuous.

LEVEL III

1. $f(x) = \begin{cases} \frac{1-\cos 4x}{8x^2}, & x\neq 0 \\ k, & x=0 \end{cases}$

Sol: Given f(x) is conti. at x = 0

$= \lim_{x\to 0} f(x) = f(0)$

$= \lim_{x\to 0} f(x) = \frac{1-\cos 4x}{8x^2} = k$

$= \lim_{x\to 0} \frac{2\sin^2 2x}{8x^2} = k \implies \lim_{x\to 0} \left(\frac{\sin^2 2x}{2x}\right)^2 = k \implies k = 1$

2. Solu.. As f(x) is cts. at x = 0 $\implies \lim_{x\to 0} f(x) = f(0)$

$$\implies f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2x+3\sin x}{3x+2\sin x} = \frac{2+3\frac{\sin x}{x}}{3+2\frac{\sin x}{x}} = 1$$

3: $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x}, & x < \frac{\pi}{2} \\ a & ,x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2}, & x > \frac{\pi}{2} \end{cases}$

Sol: Given that f(x)is continous at $x = \frac{\pi}{2}$

∴ $\text{L.H.L} = \text{R.H.L} = f\left(\frac{\pi}{2}\right)$

L.H.L $=\lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} f(x) = \lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} \frac{(1-\sin x)(1+\sin^2 x+\sin x)}{3(1-\sin x)(1+\sin x)} = \frac{3}{2*3} = \frac{1}{2}$

L.H.L $=\lim_{\substack{x\to\frac{\pi}{2}+h \\ h\to 0}} f(x) = \lim_{\substack{x\to\frac{\pi}{2}-h \\ h\to 0}} \frac{b(1-\sin x)}{(\pi-2x)^2} = \lim_{h\to 0} \frac{b(1-\sin(\frac{\pi}{2}+h)}{(\pi-2(\frac{\pi}{2}+h))^2}$

$\lim_{h\to 0} \frac{b(1-\cos h)}{(-2h)^2} = \lim_{h\to 0} \frac{b.2.\sin^2\frac{h}{2}}{4.h^2} = \lim_{h\to 0} \frac{b.1.\sin^2\frac{h}{2}}{2.4.\frac{h^2}{4}} = \frac{b}{8} * 1 = \frac{b}{8}$

$\therefore\ b = 4, a = 1/2$

4. f(x)= $\begin{cases} \dfrac{\sin x + x\cos x}{x} & , \quad when \quad x \neq 0 \\ k & , \quad when \quad x = 0. \end{cases}$

Q7: f(x)L.H.L $= \lim_{x\to 0-} f(x) = \lim_{h\to 0} \frac{\sin x}{x} + \cos x = \lim_{h\to 0} \frac{\sinh}{h} + \cosh = 1 + 1 = 2$

L.H.L $= \lim_{x\to 0+} f(x) = \lim_{h\to 0} \frac{\sin x}{x} + \cos x = 1 + 1 = 2$

$$f(0) = k$$

$\therefore\ L.H.L = R.H.L = f(0)$

$\Rightarrow \therefore k = 2$

3.Differentiation

LEVEL I

1: $f(x) = (x-1)^{2/3}$ at $x = 1$

As $f'(x) = \frac{2}{3}(x-1)^{2/3-1}.\frac{d}{dx}(x-1)$

$$= \frac{2}{3(x-1)^{1/3}}.1 \Rightarrow \frac{2}{3(x-1)^{1/3}}$$

$f'(x)$is not defined at x = 1 ,therefore f(x) is not diff. at x = 1.

2: $y = \tan^{-1}\frac{2x}{1-x^2}$

Sol: Put $x = \tan\theta$

$y = \tan^{-1}\dfrac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}(\tan 2\theta)$

$y = 2\theta \Rightarrow y = 2\tan^{-1}x$

Differentiate w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

3: $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Sol: Taking log on both sides

$\log y = \dfrac{1}{2}[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)]$

Differentiate w.r.t x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)} + \frac{2x}{(x^2+4)} - \frac{6x+4}{(3x^2+4x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{(x-3)} + \frac{2x}{(x^2+4)} - \frac{6x+4}{(3x^2+4x+5)}\right]$$

LEVEL II

1. $y = \cos\,(\log x)^2$

Sol : Differentiate w.r.t x

$$\frac{dy}{dx} = -\sin\ (\log x)^2 . \frac{d}{dx}\ (\log x)^2$$

$$= -\sin\,(\log x)^2 . 2\log x . \frac{1}{x}$$

2: $y = \tan^{-1}\left[\frac{\sqrt{x^2+1}-1}{x}\right]$

Sol: Put $x = \tan\theta$

$$y = \tan^{-1}\left[\frac{\sqrt{\tan^2\theta+1}-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2} * \cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\frac{\theta}{2}\right] = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x \ldots\ldots (i)$$

Differentiate eq (i) w.r.t x

$$\frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

3. Sol: $\frac{dy}{dx} = e^{ax}\cos bx . b + \sin bx . e^{ax} . a$

Again Differentiate w.r.t x

$$\frac{d^2y}{dx^2} = b[-e^{ax}\sin bx . b + \cos bx . e^{ax} . a] + a[e^{ax} . \cos bx . b + \ \sin bx . e^{ax} . a]$$

$$= -e^{ax}\sin bx . b^2 + ab . e^{ax}\cos x + ab . e^{ax}\cos bx + a^2 e^{ax}\sin bx$$

Put the values $\frac{dy}{dx}, \frac{d^2y}{dx^2}, y$ we get,

$\text{L.H.S} = -e^{ax}\sin bx.b^2 + ab.e^{ax}\cos x + ab.e^{ax}\cos bx + a^2e^{ax}\sin bx$

$$-2a(e^{ax}\cos bx.b + \sin bx.e^{ax}.a) + e^{ax}.\sin bx$$

On solving eq we get,

= 0 =R.H.S

Hence proved

4: $y = \frac{3at}{1+t}$

Differentiate w.r.t t

$$\frac{dy}{dt} = \frac{(1+t)3a - 3at}{(1+t)^2} = \frac{3a}{(1+t)^2}$$

$$x = \frac{2at^2}{1+t}$$

Differentiate w.r.t t

$$\frac{dx}{dt} = \frac{(1+t)4at - 2at^2}{(1+t)^2} = \frac{4at + 2at^2}{(1+t)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3a}{4at+2at^2} = \frac{3}{4t+2t^2}\ldots\ldots\ldots\ldots(i)$$

Differentiateeq(i)w.r.tx

$$\Rightarrow \frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{3}{4t+2t^2}\right) = \frac{d}{dt}\left(\frac{3}{4t+2t^2}\right).\frac{dt}{dx}$$

$$= 3\left[\frac{-(4+4t)}{(4t+2t^2)^2}\right].\frac{(1+t^2)}{4a+2at^2}$$

On solving we get,

$$= -\frac{3}{2a}\left(\frac{1+t}{2t+t^2}\right)^3$$

LEVEL III

1.Find1: $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Sol: Put $x^2 = \cos 2\theta$

$$y = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\cos^2\theta}}{\sqrt{2\sin^2\theta}-\sqrt{2\sin^2\theta}}\right] = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$y = \frac{\pi}{4}+\theta \quad = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

Differentiate w.r.t x

$$\frac{dy}{dx} = 0 + \frac{(-1).x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

2: $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{1+\cos\left(\frac{\pi}{2}-x\right)}-\sqrt{1-\cos\left(\frac{\pi}{2}-x\right)}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}+\sqrt{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}-\sqrt{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\right]$$

$$= \cot^{-1}\left[\frac{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)+\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)-\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\frac{1+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{1-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\tan\left[\frac{\pi}{4}+\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]\right]$$

$$= \cot^{-1}\left[\tan\left[\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]\right] = \cot^{-1}\left[\cot\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}$$

3: $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$

Sol: Differentiate w.r.t x

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\frac{a^2+b^2\cos^2x+2abcosx}{b^2+a^2\cos^2x+2abcosx}}}\frac{d}{dx}\left(\frac{a+bcosx}{b+acosx}\right)$$

$$=\frac{(b+acosx)[(b+acosx)[-bsinx]-(a+basinx)(-asinx)]}{\sqrt{b^2+a^2\cos^2x+2abcosx-a^2-b^2\cos^2x-2abcosx}*(b+acosx)^2}$$

$$=\frac{-bsinx(b+acosx)+(a+basinx)(asinx)}{\sqrt{b^2+a^2\cos^2x+2abcosx-a^2-b^2\cos^2x-2abcosx}*(b+acosx)}$$

$$=\frac{-bsinx(b+acosx)+(a+bcosx)(asinx)}{\sqrt{b^2+a^2\cos^2x-a^2-b^2\cos^2x}*(b+acosx)}$$

$$=\frac{(a^2-b^2)sinx}{\sqrt{b^2-a^2}sinx(b+acosx)}=\frac{-\left(\sqrt{b^2-a^2}\right)^2}{\sqrt{b^2-a^2}(b+acosx)}$$

$$=\frac{-\sqrt{b^2-a^2}}{b+acosx}$$

4. Sol: $L.H.S=\frac{1}{4\sqrt{2}}\left[\log\left(x^2+\sqrt{2}x+1\right)-\log\left(x^2-\sqrt{2}x+1\right)+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}x}{1-x^2}\right]$

$$=\frac{1}{4\sqrt{2}}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$$
$$+\frac{1}{2\sqrt{2}}\left(\frac{(1-x^2)^2}{(1-x^2)^2+2x^2}*\frac{(1-x^2)\sqrt{2}+\sqrt{2}x*2x}{(1-x^2)^2}\right)$$

$$=\frac{1}{4\sqrt{2}}\left[\frac{2x^3-2\sqrt{2}x^2+\sqrt{2}x^2+\sqrt{2}-2x^3-2\sqrt{2}x^2+\sqrt{2}x^2+\sqrt{2}}{\left(x^2+1\right)^2-\left(\sqrt{2}x\right)^2}\right]+\left[\frac{\sqrt{2}}{2\sqrt{2}}*\frac{1-x^2+2x^2}{1+x^4-2x^2+2x^2}\right]$$

$$=\frac{1}{4\sqrt{2}}\frac{-2\sqrt{2}x^2+2\sqrt{2}}{x^4+1+2x^2-2x^2}+\frac{1}{2}\left(\frac{1+x^2}{1+x^4}\right)$$

$$=\frac{1}{2}\left[\frac{1-x^2+1+x^2}{1+x^4}\right]=\frac{1}{1+x^4}=R.H.S$$

4.Logrithmic Differentiation

LEVEL I

Q1. Sol: $y=\log_7(\log x)$

$=\frac{1}{logx}\times\frac{d}{dx}(\log_7 x)\Rightarrow\frac{1}{\log x}\times\frac{1}{x\log 7}$

Q2Sol: $y=\sin(\log x)$

$y'=\cos(logx)\times\frac{d}{dx}(logx)$

$$= \frac{\cos(\log x)}{x}$$

Q3 Sol: $y = \tan^{-1}(\log x)$

$$y' = \frac{1}{1+(\log x)^2} \times \frac{d}{dx}(\log x) \Rightarrow \frac{1}{x[1+(\log x)^2]}$$

LEVEL II.

Q1 Sol: $y.\sqrt{x^2+1} = \log[\sqrt{x^2+1} - x]$

On differentiating we get,

$$\sqrt{x^2+1}y' + \frac{y(2x)}{2\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}-x} \times \frac{d}{dx}\left(\sqrt{x^2+1}-x\right)$$

$$y'\sqrt{x^2+1} + \frac{2xy}{2\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}-x}\left[\frac{2x}{2\sqrt{x^2+1}} - 1\right]$$

$$\Rightarrow (x^2+1)y' + xy = -1$$

Q2 Sol: $y = \cos(\log x)^2$

Differentiate w.r.t x

$$\frac{dy}{dx} = -\sin (\log x)^2 . \frac{d}{dx} (\log x)^2$$

$$= -\sin(\log x)^2 . 2\log x . \frac{1}{x}$$

Q3 Sol: $\frac{dy}{dx} (\cos x)^y = (\cos y)^x$

Taking log onbothsides

$y\log(\cos x) = x\log(\cos y)$

$$y'(\log\cos x) + \frac{y(\sin x)}{\cos x} = \log(\cos y) + \frac{x(-\sin y)}{\cos y}y'$$

$$y'(\log\cos x) + y\tan x = \log\cos y - xy'\tan y$$

$$y'[\log\cos x + x\tan y] = \log(\cos y) + y\tan x \Rightarrow y' = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

LEVEL III

Q1.If $x^p . y^q = (x+y)^{p+q}$, prove that $\frac{dy}{dx} = \frac{y}{x}$

Sol: Takinglogonbothsides, weget

$$p\log x + q\log y = (p+q)\log(x+y)$$

On differentiating

$$\frac{p}{x} + \frac{q}{y}y' = \frac{p+q}{x+y}[1+y'] \Rightarrow \left(\frac{q}{y} - \frac{p+q}{x+y}\right)y' = \frac{p+q}{x+y} - \frac{p}{x} \Rightarrow \frac{xq-yp}{(x+y)y}y' = \frac{qx-py}{x(x+y)}$$

$\therefore y' = \frac{y}{x}$

Q2 Sol: $y = (\log x)^{\cos x} + \frac{x^2+1}{x^2-1}$ Let $u = (\log x)^{\cos x}$ $v = \frac{x^2+1}{x^2-1}$

Taking log

$$\log u = \cos x \log(\log x)$$

Differentiating we get,

$$\frac{1}{u}\frac{du}{dx} = -\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}$$

$$\frac{du}{dx} = (\log x)^{\cos x}[\frac{\cos x}{x\log x} - \sin x \log(\log x)]$$

Taking $v = \frac{x^2+1}{x^2-1}$

$$v' = \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$\Rightarrow y' = u' + v' = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log(\log x)\right] - \frac{4x}{(x^2-1)^2}$$

Q3 Sol: $x^y = e^{x-y}$

Taking log, we get

$$y \log x = (x - y) \Rightarrow y + y\log x = x$$

$$y = \frac{x}{1 + \log x}$$

$$\Rightarrow y' = \frac{(1 + \log x) - x(\frac{1}{x})}{(1 + \log x)^2} = \frac{1 + \log x - 1}{(1 + \log x)^2} = \frac{\log x}{(\log xe)^2}$$

5 Parametric Differentiation

LEVEL-II

Q1Sol. $y = \tan x$

Differentiating we get,

$$y' = \sec^2 x$$

Again Differentiating we get,

$= y'' = 2\sec x \times \frac{d}{dx}(\sec x \tan x) = 2\sec^2 x \tan x = 2yy'$

Q2 Sol. $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and $y = a\sin\theta$

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan\frac{\theta}{2}}.\sec^2\frac{\theta}{2}.\frac{1}{2}\right) = a(-\sin\theta + \operatorname{cosec}\theta)$$

$$y = a\sin\theta \implies \frac{dy}{d\theta} = \frac{a\cos\theta}{a(-\sin\theta + \operatorname{cosec}\theta)} = \frac{\cos\theta}{\operatorname{cosec}\theta - \sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\cos\theta}{\operatorname{cosec}\theta - \sin\theta}\right)}{a(\operatorname{cosec}\theta - \sin\theta)} = \frac{\cos\theta[\operatorname{cosec}\theta \times \cot\theta - \cos\theta] + 1 - \sin^2\theta}{a(\operatorname{cosec}\theta - \sin\theta)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{2}}{a}$$

Q 3 Sol. x = $\tan\left(\frac{1}{a}\log y\right) \implies a\tan^{-1}x = \log y$

$$a\frac{1}{1+x^2} = \frac{1}{y}y' \Rightarrow \frac{a}{1+x^2} = \frac{1}{y}y' \Rightarrow ay = (1+x^2)y'$$

$$ay' = (1+x^2)y'' + 2xy'$$

$$\Rightarrow (1+x^2)y'' + (2x-1)y' = 0$$

6. Second order derivatives

LEVEL-II

Q1 Sol.If y = a cos (log x) + b sin(log x)

On differentiating we get,

$$y' = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x} \Rightarrow xy'' = -a\sin(\log x) + b\cos(\log x)$$

On differentiating we get

$$xy'' + y' = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x} \Rightarrow x^2y'' + xy' = -y$$

$$\Rightarrow x^2y'' + xy' + y = 0$$

Q2 Sol. $y=(\sin^{-1} x)^2$

On differentiating we get,

$$y' = 2\frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,y' = 2$$

On differentiating we get,

$$\Rightarrow \sqrt{1-x^2}\,y'' + \frac{-2x}{2\sqrt{1-x^2}}y' = 0 \Rightarrow (1-x^2)y'' - xy' = 0$$

Q3 Sol. $(x-a)^2 + (y-b)^2 = c^2$ for some c>0.

$(x-a)^2 + (y-b)^2 = c^2$

Differentiating w.r.t. x we get $2(x-a) + 2(y-b)y' = 0 \Rightarrow (y-b)y' = -(x-a)$

On differentiating again we get,

$$(y-b)y'' + (y')^2 = -1$$

$$\Rightarrow b = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$ is constant independent of x.

7. Mean Value Theorem

LEVEL-II

1.It is given that for the function f(x)=$x^3 - 6x^2 + px + q$ on[1,3] , Rolle's theorem holds with

c=$2+\frac{1}{\sqrt{3}}$. Find the values p and q.

Sol: $f(x) = x^3 - 6x^2 + px + q$

f(x) is cont. on [1,3] & differentiable on (1,3)

$f(1) = p + q - 5$ & $f(3) = 3p + q - 27$

As Rolle's theorem holds $f(1) = f(3)$

$\Rightarrow p = 11$ &qcantakeanyvalue

2: $f(x) = \sin x \epsilon [0, \pi]$

Sol: Since $\sin x$ is everywhere continous & differentiable

$\therefore f(x)$is cont. on $[0, \pi]$ & differentiable on (a, b)

Also $f(0) = \sin 0 = 0$, $f(\pi) = \sin \pi = 0$

$$\therefore f(0) = f(\pi)$$

Thus $f(x)$ satisfies all three conditions of Rolle's Theorem.

Now we have to show that $\exists$ $c \epsilon [0, \pi]$

s.t $f'(c) = 0$

$$f'(x) = \cos x$$

$$f'(c) = 0$$

$$\cos c = 0 = \cos \frac{\pi}{2}$$

$$c = \frac{\pi}{2} \in [0, \pi] \text{ such that } f'(c) = 0$$

3. $f(x) = \sqrt{x^2 - 4}$ in the interval [2,4]

Here $f(x)$ is cont. on $[2,4]$ & differentiable on $(2,4)$

Also$f(2) = 0$, $f(4) = \sqrt{12}$

Thus $f(x)$ satisfies all the conditions of Langrange's Theorem.

Now we have to show that $\exists$ $c \epsilon [2,4]$

s.t $\text{s.t.} f'(c) = \frac{f(4)-f(2)}{4-2}$

As $f'(x) = \frac{x}{\sqrt{x^2-4}} \Longrightarrow \frac{\sqrt{12}-0}{2} \Longrightarrow 2x = \sqrt{12}\sqrt{x^2 - 4}$

$4x^2 = 12(x^2 - 4) \Rightarrow 3x^2 - 12 = x^2 \Rightarrow 2x^2 = 12$

$x^2 = 6 \Rightarrow x = \pm\sqrt{3}$

TOPIC 5
APPLICATIONS OF DERIVATIVES

1. Rate of change

LEVEL –I

1. $\text{Diameter} = \frac{3}{2}(2x+3) \Rightarrow r = \frac{3}{4}(2x+3)$

$$V = \frac{4}{3}\pi\left[\frac{3}{4}(2x+3)\right]^3 = \frac{9}{16}\pi(2x+3)^3$$

$$\frac{dV}{dx} = \frac{9}{16}\pi.3(2x+3)^3.2 = \frac{27}{8}\pi(2x+3)^3$$

2. Let x be the side of a square

$$\frac{dx}{dt} = 4 \text{ cm/minute}$$

$$A = x^2 \Rightarrow \frac{dA}{dt} = 2x.\frac{dx}{dt} = 2\times 4x$$

$$\left.\frac{dA}{dt}\right]_{x=8} = 8\times 8 = 64 \text{ cm}^2\text{/minute}$$

3. $\frac{dr}{dt} = 0.7\text{cm/sec}$

$$C = 2\pi r$$

$$\frac{dC}{dr} = 2\pi\frac{dr}{dt} = 2\times\frac{22}{7}\times 0.7 \text{ cm/sec}$$

$$= 4.4 \text{ cm/sec}$$

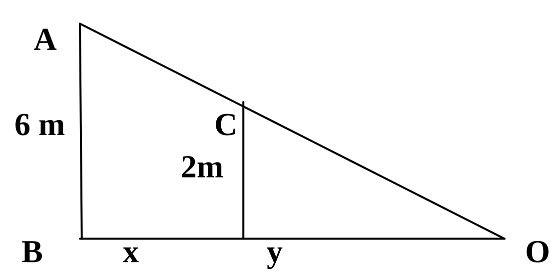
LEVEL –II

1. $y^2 = 8x \Rightarrow 2y\frac{dy}{dt} = 8\frac{dx}{dt}$

$$2y = 8 \quad \left[\frac{dy}{dt} = \frac{dx}{dt}\right] \Rightarrow y = 4,\ x = 2$$

Required point is $(2\ ,\ 4)$.

2. LetCD be the position of the man at any time t AB be the pole and OD represents



$$\frac{CD}{AB} = \frac{y}{x+y} \quad \Rightarrow \frac{2}{6} = \frac{y}{x+y}$$

$\Rightarrow \quad y = \frac{1}{2}x$

$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}x$

$\Rightarrow \quad \frac{dy}{dt} = 3 \text{ km/hr}$

3. $\frac{dx}{dt} = 3.5\text{cm/sec}, \qquad \frac{dy}{dt} = -3\text{cm/sec}$

$A = xy$

$\Rightarrow \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = -3x + 3.5y$

$\left.\frac{dA}{dt}\right]_{\substack{x=12 \\ y=8}} = -36 + 28 = -8 \text{ cm}^2/\text{sec}$

Area is decreasing at the rate of $8\text{ cm}^2/\text{sec}$

LEVEL III

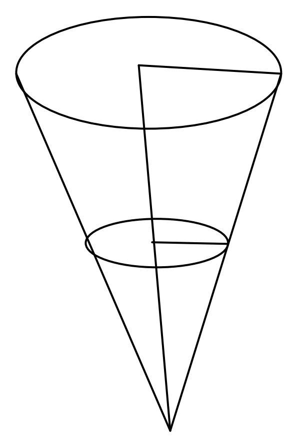
1. $6y = x^3 + 2, \quad \Rightarrow 6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$

$\Rightarrow 6 \times 8\frac{dx}{dt} = 3x^2\frac{dx}{dt} \quad \left[\because \frac{dy}{dt} = 8\frac{dx}{dt}\right]$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

$\Rightarrow y = 11, -\frac{31}{3}$

Points are $(4, 11), \left(-4, -\frac{31}{4}\right)$



2. $\frac{dV}{dt} = 5\text{ cm}^3/\text{sec}, \quad r = 10, \ h = 20$

$\frac{h}{20} = \frac{r}{10} \qquad \Rightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$

$\therefore \quad \frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} \qquad \Rightarrow \left.\frac{dh}{dt}\right]_{h=15} = \frac{4}{45\pi}\text{cm/sec}$

3. $r = 10 \text{ cm}, \ h = 50 \text{ cm}, \qquad \frac{dV}{dt} = 10 \text{ ml/sec}$

$V = \pi r^2 h = 100\pi h$

$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$

$\frac{10}{100\pi} = \frac{dh}{dt} \qquad \Rightarrow \frac{dh}{dt} = \frac{1}{10\pi}$

2. Increasing & decreasing functions

LEVEL I

1. $f(x) = x^3 - 6x^2 + 18x + 5 \qquad \Rightarrow f'(x) = 3x^2 - 12x + 18$

$= 3(x-2)^2 + 6 > 0 \ \forall \ x$

2. $f(x) = x^2 - x + 1 \qquad \Rightarrow f'(x) = 2x - 1$

$f'(x) = 0 \Rightarrow x = \frac{1}{2}$

$0 < x < \frac{1}{2} \Rightarrow f'(x) < 0 \qquad \Rightarrow f(x) \text{ is } \downarrow$

$\frac{1}{2} < x < 1 \Rightarrow f'(x) > 0 \qquad \Rightarrow f(x) \text{ is } \uparrow$

Function is neither increasing nor decreasing in $(0, 1)$

3. $f'(x) = \cos x + \sin x = 0 \qquad \Rightarrow \tan x = -1$

$\Rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$

For $0 < x < \frac{3\pi}{4}, \qquad f'(x) > 0 \qquad \Rightarrow f(x) \text{ is } \uparrow$

For $\frac{3\pi}{4} < x < \frac{7\pi}{4}, \qquad f'(x) < 0 \qquad \Rightarrow f(x) \text{ is } \downarrow$

For $\frac{7\pi}{4} < x < \pi, \qquad f'(x) > 0 \qquad \Rightarrow f(x) \text{ is } \uparrow$

LEVEL II

1. $f(x) = \cos x, \quad \Rightarrow f'(x) = -\sin x$

For $0 < x < \pi, \qquad f'(x) < 0 \qquad \Rightarrow f(x) \text{ is } \downarrow$

2. $f'(x) = \frac{x\cos x - \sin x}{x^2} < 0$

as $x\cos x - \sin x < 0, \ \forall \ x \in (o, \pi/2)$ and $x^2 > 0$

$\Rightarrow \qquad f(x) = \frac{\sin x}{x}$ is strictly decreasing in $(o, \pi/2)$

3. $f'(x) = \frac{1-\log x}{x^2} = 0 \quad \Rightarrow 1-\log x = 0 \quad \Rightarrow x = e$

$\therefore f(x)$ is $\uparrow$ in $(0, e)$ and $\downarrow$ in (e, ∞)

LEVEL III

1. $f'(x) = 4x - \frac{1}{x} = 0 \quad \Rightarrow 4x^2 = 1 \quad \Rightarrow x = \pm\frac{1}{2}, \; x \neq 0$

For $x \in (-\infty, -1/2)$, $f'(x) ,< 0$

For $x \in (-½, 0)$, $f'(x) > 0$

For $x \in (0, 1/2)$ $f'(x) ,< 0$

For $x \in$ For $x \in (½, \infty)$, $f'(x) > 0$

$\therefore$ f(x) is $\uparrow$ in $(-½, 0) \cup (½, \infty)$ and $\downarrow$ in $(-\infty, -1/2) \cup (0, 1/2)$

2. $\frac{dy}{d\theta} = 4\left[\frac{2\cos\theta + 1}{(2+\cos\theta)^2}\right] - 1 = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$

$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$ is positive $\forall\, \theta$

sign of $\frac{dy}{dx}$ depends upon $\cos\theta$

Hence function is $\uparrow$ in $\left(0, \frac{\pi}{2}\right)$

3. Tangents &Normals

LEVEL-I

1. $3x^2 - y^2 = 8 \quad \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$

Slope of normal $= -\frac{y}{3x}$

Slope of line $= -\frac{1}{3}$

$\Rightarrow -\frac{y}{3x} = -\frac{1}{3} \Rightarrow x = y$

$\therefore\; x = y = \pm 2$

Equation of normal is $y - 2 = -\frac{1}{3}(x-2) \quad \Rightarrow x + 3y - 8 = 0$

and $y + 2 = -\frac{1}{3}(x+2) \quad \Rightarrow x + 3y + 8 = 0$

2. $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$2x = x \Rightarrow x = 0,\ y = 0$

$\therefore$ point is $(0, 0)$

3. $x^2 + y^2 - 2x - 4y + 1 = 0 \qquad \Rightarrow 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{y-2}$

Tangent is parallel to x – axis $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{1-x}{y-2} = 0$

$\Rightarrow \quad x = 1$ and $y = 0,4$

$\therefore$ Points are $(1,0)$ and $(1, 4)$

LEVEL-II

1. $ay^2 = x^3 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}$

Slope of tangent at $(am^2 \quad am^3) = \frac{3}{2}m$

Slope of normal at $(am^2 \quad am^3) = -\frac{2}{3}m$

Equation of normal is $y - am^3 = -\frac{2}{3m}\left(x - am^2\right)$

i.e. $2x + 3my - 3am^4 - 2am^2 = 0$

2. $y = 2x^2 + 3x + 18$ ....(1)

$\Rightarrow \frac{dy}{dx} = 4x + 3$

Let the point be (x_1, y_1)

Slope at this point is $4x_1 + 3$ and equation of tangent is

$y - y_1 = (4x_1 + 3)(x - x_1)$

$\because$ it passes through $(0, 0)$

$\Rightarrow \quad y_1 = x_1(4x_1 + 3)$................(2)

Also (x_1, y_1) lies on $y = 2x^2 + 3x + 18$

$\Rightarrow \quad y_1 = 2x_1^2 + 3x_1 + 18$.............(3)

From (2) and (3) $x_1 = \pm 3$ and $y_1 = 45, 27$

Points are $(3, 45)$ and $(-3, 27)$

3. $y = x^3 + 2x + 6 \qquad \Rightarrow \frac{dy}{dx} = 3x^2 + 2$

Slope of normal is $-\frac{1}{3x^2+2}$

Slope of line is $-\frac{1}{14}$

$\therefore \quad -\frac{1}{14} = -\frac{1}{3x^2+2} \qquad \Rightarrow x = \pm 2 \Rightarrow y = 18, -6$

Points are $(2, 18)$ $(-2, -6)$

LEVEL- III

1. $y = \sqrt{5x-3} - 2 \qquad \Rightarrow \frac{dy}{dx} = \frac{5}{2\sqrt{5x-3}}$

Slope of line is 2

$\therefore \quad \frac{5}{2\sqrt{5x-3}} = 2 \qquad \Rightarrow x = \frac{73}{80}, \; y = -\frac{3}{4}$

Point is $\left(\frac{73}{80}, -\frac{3}{4}\right)$

Equation of tangent is $y + \frac{3}{4} = 2\left(x - \frac{73}{80}\right) \Rightarrow 80x - 40y - 103 = 0$

2. Solving $x^2 + y^2 - 2x = 0$ ....(i)

and $x^2 + y^2 - 2y = 0$ ....(ii) we get $x = y$

Differentiating (i) w. r. t. x, $2x + 2y\frac{dy}{dx} - 2 = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{1-x}{y} = \infty$ at $(0, 0)$

$\Rightarrow$ tangent is parallel to $y-$axis

Differentiating (ii) w. r. t. x, $2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y-1} = 0$ .at $(0, 0)$.

$\Rightarrow$ tangent is parallel to $x-$axis

$\therefore$ curves are orthogonal

3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating w.r.t. x $\quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \qquad \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$..........(i)

$xy = c^2$

Differentiating w.r.t. x $\quad x\frac{dy}{dx} + y = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$............(ii)

Since curves cut orthogonally $\Rightarrow$ product of slopes $= -1$

$$\Rightarrow \frac{b^2}{a^2}\frac{x}{y} \times -\frac{y}{x} = -1 \qquad \Rightarrow b^2 = a^2$$

4. Approximations

LEVEL-I

1 Sol: $y = \sqrt{25.3}$

Let $x = 25, \Delta x = 0.3, y = \sqrt{x}$

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{25.3} - 5 \ldots \ldots \ldots \ldots \ldots \text{(i)}$

To find Δy, $\Delta y = \left(\frac{dy}{dx}\right)\Delta x$ ($\because$ Δx is approx equal to dy)

$\Delta y = \left(\frac{1}{2\sqrt{x}}\right)(0.3) \Longrightarrow \Delta y = \left(\frac{1}{2 \times 5}\right) \times 0.3 = 0.03$

From (i) $\sqrt{25.3} = 5 + 0.03 = 5.03$

Q2 Sol: $y = (66)^{\frac{1}{3}}$

Let $y = (x)^{\frac{1}{3}}; x = 64, \Delta x = 2$

$\Delta y = (66)^{\frac{1}{3}} - 4$

$\therefore (66)^{\frac{1}{3}} = 4 + \Delta y \ldots \text{(i)}$

$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$

$\Longrightarrow \Delta y = \left(\frac{1}{3x^{\frac{2}{3}}}\right)\Delta x = \left(\frac{1}{3(64)^{\frac{2}{3}}}\right) \times 2 = \frac{2}{3X16} = \frac{2}{48} = 0.042$

From (i) $(66)^{\frac{1}{3}} = 4 + 0.042 = 4.042$

Q3 Sol: $y = \sqrt{0.082}$

$y = \sqrt{x}; \quad x = 0.09 \ \& \Delta x = -0.008$

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{0.082} - 0.3 \ldots \ldots \ldots \ldots \ldots \text{(i)}$

$\sqrt{0.082} = 0.3 + \Delta y$ ... (i)

To find $\Delta y, \Delta y = \left(\frac{dy}{dx}\right)\Delta x$

$$\Rightarrow \frac{1}{2\sqrt{x}} \times (-0.008) = -\frac{0.008}{2\sqrt{0.09}} = \frac{-0.004}{0.3} = -\frac{4}{3} \times \frac{1}{100} = -0.0133$$

$\therefore \sqrt{.082} = 0.3 - 0.0133 = 0.2867$

5 Maxima & Minima

LEVEL I

1. $f(x) = 3 - 2\sin x$

$\because -1 \le \sin x \le 1 \quad \Rightarrow 2 \ge -2\sin x \ge -2 \quad \Rightarrow 3 + 2 \ge 3 - 2\sin x \ge 3 - 2$

$\Rightarrow 1 \le 3 - 2\sin x \le 5$

$\therefore$ Min value of f(x) is 1 and max value is 5.

2. $f(x) = x^3 + x^2 + x + 1 \quad \Rightarrow f'(x) = 3x^2 + 2x + 1$

$f'(x) = 3x^2 + 2x + 1 = 0$ has no real roots

Hence f(x) has neither max. nor min. value

3. Let $x + y = 24$ and $P = xy$

$P = x(24 - x) = 24x - x^2 \quad \Rightarrow \frac{dP}{dx} = 24 - 2x$

$\frac{dP}{dx} = 0 \Rightarrow x = 12$

$\frac{d^2P}{dx^2} = -2 < 0$ when $x = 12, y = 12$

$\therefore$ P is maximum when $x = 12, y = 12$ and hence required numbers are 12,12

LEVEL II

1. In right triangle $x^2 + y^2 = h^2$

Area $= \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$

$$\frac{dA}{dx} = \frac{1}{2}\left[\frac{x(-2x)}{2\sqrt{h^2 - x^2}} + \sqrt{h^2 - x^2}\right] = \frac{1}{2}\left[\frac{h^2 - 2x^2}{\sqrt{h^2 - x^2}}\right]$$

For max. or min. $\frac{dA}{dx} = 0 \quad \Rightarrow h^2 - 2x^2 = 0 \quad \Rightarrow x = \frac{h}{\sqrt{2}}, y = \frac{h}{\sqrt{2}}$

$$\frac{d^2A}{dx^2}=\frac{1}{2}\left[\frac{\sqrt{h^2-x^2}.(-4x)-(h^2-2x^2)\times\frac{-2x}{2\sqrt{h^2-x^2}}}{h^2-x^2}\right]$$

$$=\frac{1}{2}\left[\frac{-4x(h^2-x^2)+x(h^2-2x^2)}{(h^2-2x^2)^{3/2}}\right]=\frac{1}{2}\left[\frac{-4x(x^2)+0}{x^3}\right]\quad\left[\because 2x^2=h^2\right]$$

$$=-2<0$$

$\therefore$ Area is maximum when triangle is issosceles

2. Let one piece $= x$

$\therefore$ other piece $= 28-x$

Let x unit be made into a circle and that $(28-x)$ be made into a square

$\therefore 2\pi r = x \qquad \Rightarrow r=\frac{x}{2\pi}$

Area of circle $=\pi\left(\frac{x}{2\pi}\right)^2=\frac{x^2}{4\pi}$

Side of the square $=\frac{28-x}{4}$

Area of the square $=\left(\frac{28-x}{4}\right)^2$

Total area $A=\frac{x^2}{4\pi}+\left(\frac{28-x}{4}\right)^2$

$$\frac{dA}{dx}=\frac{x}{2\pi}-\frac{28-x}{8}$$

For max. or min. $\frac{dA}{dx}=0 \qquad \Rightarrow \frac{x}{2\pi}=\frac{28-x}{8}\Rightarrow x=\frac{28\pi}{4+\pi}$

$$\frac{d^2A}{dx^2}=\frac{1}{2\pi}+\frac{1}{8}>0$$

$\therefore$ A is min when $x=\frac{28\pi}{4+\pi}$ and other part $=28-\frac{28\pi}{4+\pi}=\frac{112}{4+\pi}$

3. Let r units be the radius of the semicircle and h units be the side AD of the rectangle ABCD.Since the perimeter is 10 ,

$$2h+2r+\pi r=10$$

$2h = 10-2r-\pi r h=\frac{1}{2}(10-2r-\pi r)$

Area$= 2rh+\frac{1}{2}\pi r^2$

$=2r\times\frac{1}{2}\left(10-\ 2r-\pi r\right)+\frac{1}{2}\pi r^2$

$=10r - 2\,r^2 - \frac{1}{2}\pi r^2$

$\frac{dA}{dx}$ = 10-4r-πr

$now\ \ max\ value\ \frac{dA}{dx}=0$

➔ 10-4r-πr = 0

r (4+π)=10

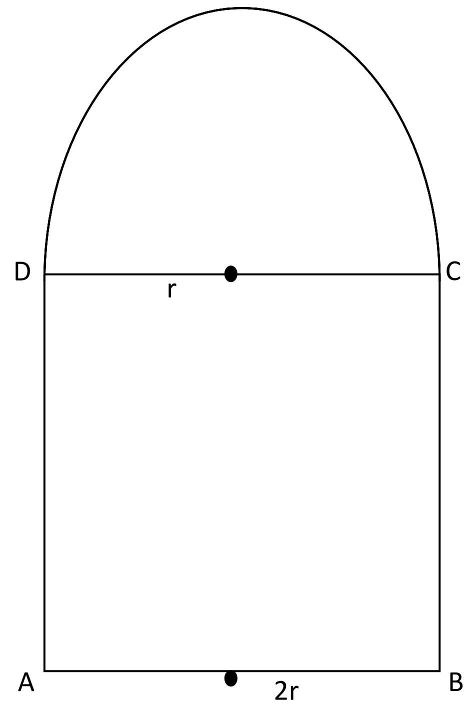
r=$\frac{10}{(\pi+4)}$

$\frac{d^2(A)}{dr^2}$= - 4- π =- $(\pi+4)$ <0

A has local maximum when r=$\frac{10}{(\pi+4)}$

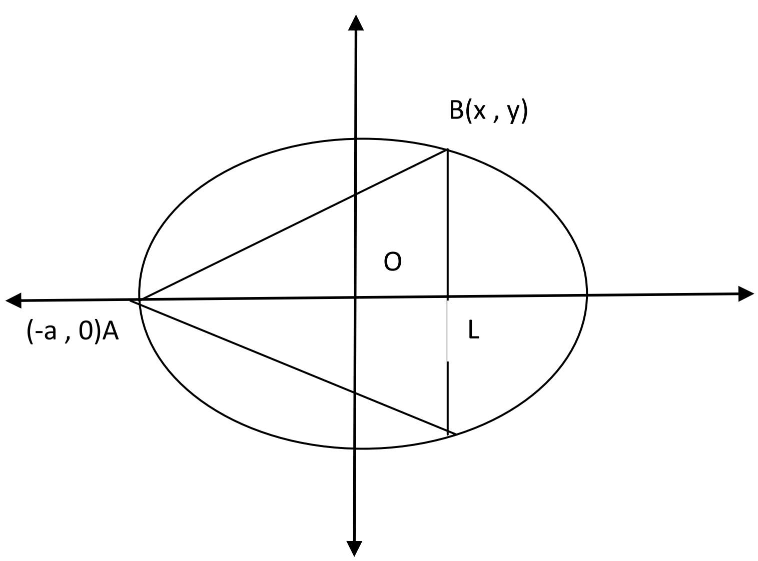
Length = 2r = $\frac{20}{(\pi+4)}$m

h=$\frac{1}{2}\left(10-(2+\pi).\frac{10}{(\pi+4)}\right)$=$\frac{10}{(\pi+4)}$



LEVEL III

1.



1.Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and B be (x , y)

Then $BC=2y$ and $AL=a+x$

$$A=\frac{1}{2}.2y.(a+x)=(a+x)y=(a+x).\frac{b}{a}\sqrt{a^2-x^2}$$

$$=\frac{b}{a}(a+x)^{3/2}(a-x)^{1/2}.............(i)$$

$$\frac{dA}{dx}=\frac{b}{a}\left[(a+x)^{3/2}.\frac{1}{2\sqrt{a-x}}(-1)+(a-x)^{1/2}.\frac{3}{2}(a+x)^{1/2}\right]$$

$$=\frac{b}{a}\sqrt{a+x}\left[\frac{a-2x}{\sqrt{a-x}}\right]$$

For max.area $\frac{dA}{dx}=0 \quad \Rightarrow x=\frac{a}{2}$

$$\frac{d^2A}{dx^2}<0$$

$$\therefore A_{Max}=\frac{b}{a}\left(\frac{3a}{2}\right)^{3/2}\left(\frac{a}{2}\right)^{1/2}=\frac{3\sqrt{3}}{4}\text{ ab sq.units}$$

2.Let x be the side of the square base & *let y be the height of the square.*

S.A of box= $lb+2(lh+bh)=x^2+2(xy+xy)=x^2+4xy$

S.A of box = Area of cardboard

$x^2+4xy=c^2$

$4xy=c^2-x^2$

$$y=\frac{c^2-x^2}{4x}\ldots\ldots.(*)$$

Volume of box= x^2y

$$V=x^2y=x^2\left(\frac{c^2-x^2}{4x}\right)$$

$$V=\frac{xc^2-x^3}{4}$$

$$\frac{dV}{dx}=\frac{1}{4}(c^2-3x^2)$$

For max. Volume, $\frac{dV}{dx}=0$

$$\frac{1}{4}(c^2 - 3x^2) = 0$$

$$c^2 - 3x^2 = 0$$

$$c^2 = 3x^2$$

$$x^2 = \frac{c^2}{3}$$

$$From(*), y = \frac{c^2 - \frac{c^2}{3}}{4\left(\frac{c}{\sqrt{3}}\right)}$$

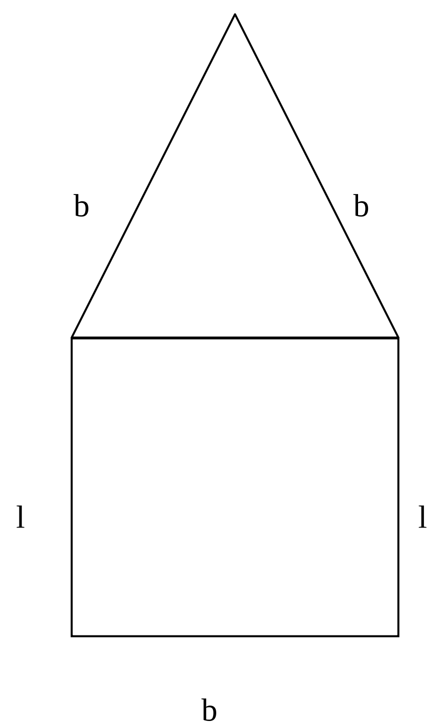
$$y = \frac{2\left(\frac{c^2}{3}\right)}{\frac{4c}{\sqrt{3}}} = \frac{2c^2}{3}.\frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = -\frac{3}{2}x < 0$$

$\therefore$ V is max. when $x = \frac{c}{\sqrt{3}}$ & $y = \frac{c}{2\sqrt{3}}$

$$\text{Max. volume} = x^2y = \frac{c^2}{3}.\frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \text{cu units}$$

3. Perimeter of window =12cm



$$2(l + b) + 2b = 12$$

$$2l + 2b + 2b = 12$$

$$2l + 4b = 12$$

$$l = \frac{12-4b}{2} = \frac{4(3-b)}{2} = 2(3-b) = 6 - 2b$$

$$A = \text{Area of window} = lb + \frac{\sqrt{3}}{4}b^2 = (6-2b)b + \frac{\sqrt{3}}{4}b^2$$

$$= 6b - 2b^2 + \frac{\sqrt{3}}{4}b^2 = 6b + \frac{-8b^2 + b^2\sqrt{3}}{4} = 6b + \frac{b^2(\sqrt{3}-8)}{4}$$

$$\frac{dA}{db} = 6 + \frac{2(\sqrt{3}-8)b}{4} = 6 + \frac{(\sqrt{3}-8)b}{2}$$

$$\frac{dA}{db} = 0$$

$$6 + \frac{(\sqrt{3} - 8)\text{b}}{2} = 0$$

$$\frac{12 + (\sqrt{3} - 8)\text{b}}{2} = 0$$

$$(\sqrt{3} - 8)\text{b} = -12$$

$$\text{b} = \frac{-12}{(\sqrt{3} - 8)} = \frac{-12}{8 - \sqrt{3}}$$

$$\frac{d^2A}{db^2} = \frac{\sqrt{3} - 8}{2} < 0$$

$\therefore$ Area is maximum when $\text{b} = \dfrac{12}{8 - \sqrt{3}}$

$$l = 6 - 2b = 6 - 2\left(\frac{12}{8 - \sqrt{3}}\right)$$

$$= 6 - \frac{24}{8 - \sqrt{3}}$$

$$= \frac{48 - 6\sqrt{3} - 24}{8 - \sqrt{3}} = \frac{6(4 - \sqrt{3})}{8 - \sqrt{3}}$$

TOPIC-6

INDEFINITE & DEFINITE INTEGRALS

(i) Integration by substitution

LEVEL I

Q1. $\int \frac{\sec^2(\log x)}{x} dx$

Sol. $Put log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore \int sec^2 t dt = tan t + C$

$= \tan(\log x) + C$

Q2. $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

Sol: put $m tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = \frac{dt}{m}$

$= \frac{1}{m} \int e^t \, dt = e^t + C$

$= e^{m \, tan^{-1} x} + C$

$Q3.$ $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$

Sol. Put $sin^{-1} x = t \Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

$= \int e^t \, dt = e^t + C = e^{sin^{-1} x} + C$

LEVEL II

Q1.$\int \frac{dx}{\sqrt{x}+x} = \int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

Sol. Put $\sqrt{x} = t \Rightarrow \frac{dx}{\sqrt{x}} = 2dt$

$= \int \frac{2dt}{1+t} = 2 \, log(1+t) + c$

$= 2 \log(1+\sqrt{x}) + c$

Q2. $\int \frac{dx}{x\sqrt{x^6-1}}$

Sol. $\int \frac{dx}{x\sqrt{(x^3)^2-1}}$ put $x^3 = t, \Longrightarrow 3x^2dx = dt$

$dx = \frac{dt}{3x^2}$

$\therefore \int \frac{dt}{3x^3\sqrt{(x^3)^2-1}}$

$$= \int \frac{dt}{3t\sqrt{t^2-1}} = \frac{1}{3}sec^{-1}\,t + c \;= \frac{1}{3}sec^{-1}\,x^3 + c$$

$Q3. \int \frac{dx}{e^x - 1}$

$Sol. put e^x - 1 = t \;\Longrightarrow e^x dx = dt \;\Longrightarrow dx = \frac{dt}{e^x} = \frac{dt}{t+1}$

$\therefore \int \frac{dt}{t(t+1)} = \;\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$

$$= logt - log(1+t) + c$$

$$= log\left|\frac{t}{1+t}\right| + c \;=\; log\left|\frac{e^x - 1}{e^x}\right| + c$$

LEVEL III

Q1.$\int \frac{\sqrt{tanx}}{sinxcosx}dx$

Sol. $\int \frac{tanxdx}{sinx\,cos\,x\sqrt{tanx}}$

$= \int \frac{\frac{sinx}{cosx}dx}{sinxcosx\sqrt{tanx}}$

$= \int \frac{dx}{cos^2x\sqrt{tanx}}$

$= \;\int \frac{sec^2x}{\sqrt{tanx}}dx$

$puttanx = t \;\Longrightarrow sec^2xdx = dt$

$= \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{tanx} + \;C$

Q2. $\int \frac{\tan x}{\sec x + \cos x} dx$

Sol. $\int \frac{\frac{sinx}{cosx}}{\frac{1}{cosx}+cosx} dx$

$= \int \frac{sinxdx}{1+cos^2x}$

$put\ cos\,x = t \implies -sinxdx = dt$

$$\therefore \int \frac{-dt}{1+t^2} = -tan^{-1}\,t + C = -tan^{-1}(cosx) + C$$

Q3. $\int \frac{dx}{sinxcos^3x}$

Sol. Multiplying Nr. & Dr. both by sec^4x , $weget$

$= \int \frac{sec^4x}{tanx} dx = \int \frac{sec^2xsec^2x}{tanx} dx$

$= \int \frac{(1+tan^2x)sec^2x}{tanx} dx$

$put\ \ tanx = t \implies sec^2xdx = dt$

$\therefore \int \frac{(1+t^2)}{t} dt = \int \left(\frac{1}{t} + t\right)$dt $= logt + \frac{t^2}{2} + C$

$= logtanx + \frac{tan^2x}{2}$ + C

(ii) Application of trigonometric function in integrals

LEVEL I

Q1.$\int sin^3x\,dx$

Sol.$Since\ \ sin3x = 3sinx - 4sin^3x$

$\implies sin^3x = \frac{3sinx - sin3x}{4}$

$\therefore \int sin^3x\,dx = \int \frac{3sinx - sin3x}{4} dx$

$= -\frac{3}{4}cosx + \frac{cos3x}{12} + C$

Q2.$\int cos^2 3x dx$

Sol.$\int \frac{(1+cos6x)}{2}dx$ $[\therefore\ cos^2 3x = \frac{1+cos6x}{2}]$

$=\int \left(\frac{1}{2} + \frac{cos6x}{2}\right)dx = \frac{1}{2}x + \frac{sin6x}{12} + C$

Q3. $\int cosxcos2xcos3xdx$

Sol. $=\frac{1}{2}\int cos2x(2cosxcos3x)dx$

$=\frac{1}{2}\int cos2x(cos4x + cos2x)dx$

$= \frac{1}{2}\int cos2xcos4xdx + \frac{1}{2}\int cos^2 2xdx$

$= \frac{1}{4}\int 2cos2xcos4xdx + \frac{1}{4}\int 2cos^2 2xdx$

$= \frac{1}{4}\int (cos6x + cos2x)dx + \frac{1}{4}\int (1 + cos4x)\,dx$

$= \frac{1}{4}[\frac{sin6x}{6} + \frac{sin2x}{2} + x + \frac{sin4x}{4}] + c$

LEVEL II

Q1. $\int sec^4 xtanxdx$

Sol. $= \int sec^3 x\,(secxtanx)dx$

$Put\ \ secx\ = t\ \Rightarrow secxtanxdx = dt$

$= \int t^3 dt = \frac{t^4}{4} + c$

$= \frac{sec^4 x}{4} + c$

Q2.$\int \frac{sin4x}{sinx}dx$

Sol. $= \int \frac{2sin2xcos2x}{sinx}dx$

$= \int \frac{4sinxcosxcos2x}{sinx}dx$

$=\int 4cosxcos2xdx$

$= 2\int (2cosxcos2x)dx$

$= 2\int (cos3x + cosx)dx$

$= \frac{2}{3}sin3x + 2sinx + c$

LEVEL III

Q1.$\int cos^5xdx$

Sol. $= \int cos^4xcosx\, dx$

$= \int (1 - sin^2x)^2 \cos xdx$

$put\ \ sinx = t\ \ \Longrightarrow cosxdx = dt$

$$\therefore \int (1 - t^2)^2dt \Longrightarrow \int (1 + t^4 - 2t^2)dt$$

$= t + \frac{t^5}{5} - \frac{2}{3}t^3 + c$

$= sinx + \frac{sin^5x}{5} - \frac{2}{3}sin^3x + c$

Q2. $\int sin^2x\, cos^3x$dx

Sol. $\int sin^2x(1 - sin^2x)cosxdx$

$put\ \ sinx = t \Longrightarrow cosxdx = dt$

$= \int t^2(1 - t^2)dt$

$= \int (t^2 - t^4)dt = \frac{t^3}{3} - \frac{t^5}{5} + c$

$= \frac{sin^3x}{3} - \frac{sin^5x}{5} + c$

(iii) Integration using standard results

LEVEL I

Q1.$\int \frac{dx}{\sqrt{4x^2-9}}$

Sol. $= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\frac{9}{4}}}$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$$

$$= \frac{1}{2}log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right|+c$$

Q2.$\int \frac{dx}{x^2+2x+10}$

Sol.∴ $x^2+2x+10 = x^2+2.x.1+1^2+10-1^2$

$$= (x+1)^2+3^2$$

.∴ $\int \frac{dx}{(x+1)^2+3^2} = \frac{1}{3}tan^{-1}\left(\frac{x+1}{3}\right)+c$

Q3. $\int \frac{dx}{9x^2+12x+13}$

Sol. .∴ $9x^2+12x+13 = 9(x^2+\frac{4}{3}x+\frac{13}{9})$

$$= 9(x^2+2.x.\frac{2}{3}+\frac{4}{9}+\frac{13}{9}-\frac{4}{9})$$

$$= 9(\left(x+\frac{2}{3}\right)^2+1^2)$$

$$∴ \int \frac{dx}{9(\left(x+\frac{2}{3}\right)^2+1^2)} = \frac{1}{9}\int \frac{dx}{\left(x+\frac{2}{3}\right)^2+1^2}$$

$=\frac{1}{9}tan^{-1}\left(\frac{x+2/3}{1}\right)=\frac{1}{9}tan^{-1}\left(\frac{3x+2}{3}\right)+c$

LEVEL II

Q1.$\int \frac{xdx}{x^4+x^2+1}$

Sol. Put $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = dt/2$

$$\therefore \int \frac{dt/2}{t^2+t+1}$$

$$\therefore\ t^2+t+1 = t^2+2.t.\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1$$

$$= \left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}} tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+c = \frac{1}{\sqrt{3}} tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$$

Q2 $\int \frac{cosx\,dx}{sin^2x+4sinx+5}$

Sol. Put $sinx = t \implies cosxdx = dt$

$$\implies \int \frac{dt}{t^2+4t+5}$$

$$\therefore t^2+4t+5 = (t+2)^2+1$$

$$\implies \int \frac{dt}{(t+2)^2+1} = tan^{-1}(t+2)+c$$

$$= tan^{-1}(sinx+2)+c$$

Q3.$\int \frac{dx}{\sqrt{7-6x-x^2}}$

Sol. $\therefore 7-6x-x^2 = -(x^2+6x-7)$

$$= -[(x+3)^2-16] = 4^2-(x+3)^2$$

$$\therefore \int \frac{dx}{\sqrt{4^2-(x+3)^2}} = sin^{-1}\left(\frac{x+3}{4}\right)+c$$

LEVEL III

Q1.$\int \frac{2x}{\sqrt{1-x^2-x^4}}dx$

Sol. $putx^2 = t \implies 2xdx = dt$

$\therefore \int \frac{dx}{\sqrt{1-t-t^2}}$

Now, $1-t-t^2 = -(t^2+t-1) = -\{\left(t^2+t+\frac{1}{4}\right)-\left(\frac{1}{4}+1\right)\}$

$= -\{\left(t+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 - \left(t+\frac{1}{2}\right)^2$

$=\int \frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(t+\frac{1}{2}\right)^2}} = sin^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)$+c $= sin^{-1}\left(\frac{2t+1}{\sqrt{5}}\right)+c$

$= sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right)+c$

Q2 $\int \frac{x^2+x+1}{x^2-x+1}dx$

Sol. $\int \frac{x^2-x+1+2x}{x^2-x+1}dx$

$=\int \left(1+\frac{2x}{x^2-x+1}\right)dx$

$= \int \left(1+\frac{2x-1+1}{x^2-x+1}\right)dx$

$= \int \left(1+\frac{2x-1}{x^2-x+1}+\frac{1}{x^2-x+1}\right)dx$

$\therefore x^2-x+1 = x^2-2.x.\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1 = (x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2$

$= \int \left(1+\frac{2x-1}{x^2-x+1}+\frac{1}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)dx$

$= x+log|x^2-x+1|+\frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

Q3 $\int \frac{x+2}{\sqrt{x^2+5x+6}}$

Sol. Let $x+2 = A\frac{d}{dx}(x^2+5x+6)+B$

$\Longrightarrow x + 2 = A(2x + 5) + B$

$\Longrightarrow \; x + 2 = 2Ax + 5A + B$

$\therefore 2A = 1 \qquad\qquad ; 5A + B = 2$

$\Longrightarrow A = \frac{1}{2}\,, B = -\frac{1}{2}$

$$\Longrightarrow \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx = \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x + \frac{5}{2} + \sqrt{x^2+5x+6}\right| + C$

Q4 $\int \sqrt{\frac{1-x}{1+x}}dx$

Sol. $\int \sqrt{\frac{1-x}{1+x} X \frac{1-x}{1-x}}dx = \int \frac{1-x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int \frac{(-2x)}{\sqrt{1-x^2}}dx$

$$= sin^{-1} x + \sqrt{1-x^2} + C$$

Q5 $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

Sol. $(x-5)(x-4) = x^2 - 9x + 20$

$let \; 6x + 7 = A\frac{d}{dx}(x^2 - 9x + 20) + B = A(2x - 9) + B$

$\Longrightarrow 6x + 7 = 2Ax + (B - 9A)$

$\Longrightarrow A = 3\,; B = 34$

$\Longrightarrow \int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$

$= 3\int \frac{(2x-9)}{\sqrt{x^2-9x+20}}dx + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$

I=$3I_1$ +$34I_2$ ……………………(1)

$$let I_1 = \int \frac{(2x-9)}{\sqrt{x^2-9x+20}}dx$$

Put $x^2 - 9x + 20 = t \implies (2x-9)dx = dt$

$\implies \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{x^2-9x+20} + C$

$$I_2 = \int \frac{dx}{\sqrt{x^2-9x+20}}$$

$$x^2 - 9x + 20 = \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$=\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

(1) $\implies I = 6\sqrt{x^2-9x+20} + 34\, log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

(iv) Integration using Partial Fraction

LEVEL I

Q1. $\int \frac{2x+1}{(x+1)(x-1)}dx$

$Sol. Let \frac{2x+1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$\therefore 2x + 1 = A(x-1) + B(x+1) = (A+B)x + (B-A)$

$\implies A + B = 2\ ; B - A = 1$

$\implies A = \frac{1}{2}\ ; B = \frac{3}{2}$

$$\int \frac{2x+1}{(x+1)(x-1)}dx = \frac{1}{2}\int \frac{dx}{(x+1)} + \frac{3}{2}\int \frac{dx}{(x-1)}$$

$= \frac{1}{2}log|x+1| + \frac{3}{2}log|x-1| + C$

$Q2. \int \frac{x^2}{(x-1)(x-2)(x-3)}dx$

Sol. $Let \frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$\implies x^2 = A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$$\Rightarrow x^2 = (A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C)$$

$$\Rightarrow A+B+C = 1\ ;\ -5A-4B-3C = 0\ ;\ 6A+3B+2C = 0$$

$$\Rightarrow A = \frac{1}{2}; B = -4\ ; C = \frac{9}{2}$$

$$\Rightarrow \int \frac{x^2}{(x-1)(x-2)(x-3)}dx = \frac{1}{2}\int \frac{dx}{(x-1)} - 4\int \frac{dx}{(x-2)} + \frac{9}{2}\int \frac{dx}{(x-3)}$$

$$= \frac{1}{2}log|x-1| - 4\, log|x-2| + \frac{9}{2}log|x-3| + C$$

Q3 $\int \frac{3x-2}{(x+1)^2(x+3)}dx$

Sol. $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$

$$\Rightarrow 3x-2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\Rightarrow A+C = 0\ ; 4A+B+2C = 3\ ; 3A+3B+C = -2$$

$$\Rightarrow A = \frac{11}{4}\ ; B = -\frac{5}{2}\ ; C = -\frac{11}{4}$$

$$\int \frac{3x-2}{(x+1)^2(x+3)}dx = \frac{11}{4}\int \frac{dx}{(x+1)} - \frac{5}{2}\int \frac{dx}{(x+1)^2} - \frac{11}{4}\int \frac{dx}{(x+3)}$$

$$=\frac{11}{4}log|x+1| + \frac{5}{2}\left(\frac{1}{x+1}\right) - \frac{11}{4}log|x+3| + C$$

LEVEL II

Q1. $\int \frac{x^2+2x+8}{(x-1)(x-2)}dx=$

Sol. Let $\frac{x^2+2x+8}{(x-1)(x-2)} = 1 + \frac{5x+6}{x^2-3x+2}$

$$\therefore \int \frac{x^2+2x+8}{(x-1)(x-2)}dx = \int \left(1 + \frac{5x+6}{x^2-3x+2}\right)dx$$

$= x + \int \left(\frac{5x+6}{x^2-3x+2}\right)dx$ …………….(1)

$I = x + I_1$

Where $I_1 = \int \left(\frac{5x+6}{x^2-3x+2}\right)dx$

$Let\ 5x+6 = A\frac{d}{dx}(x^2-3x+2)+B$

$\Rightarrow 5x+6 = A(2x-3)+B$

$\Rightarrow 5x+6 = 2Ax+(B-3A)$

$\Rightarrow A{=}\frac{5}{2}\ ; B{=}\frac{27}{2}$

$\therefore I_1 = \int \frac{\frac{5}{2}(x^2-3x+2)+\frac{27}{2}}{x^2-3x+2}dx$

$= \frac{5}{2}\int dx{+}\frac{27}{2}\int\frac{dx}{x^2-3x+2} = \frac{5}{2}x+\frac{27}{2}\int\frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$

$$= \frac{5}{2}x+\frac{27}{2}\log\left|\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}\right|+C$$

$$= \frac{5}{2}x+\frac{27}{2}\log\left|\frac{x-2}{x-1}\right|+C$$

Q2. $\int\frac{x^2+x+1}{x^2(x+2)}dx$

Sol. Let $\frac{x^2+x+1}{x^2(x+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$

$= A(x+2)x+B(x+2)+Cx^2$

$x^2+x+1 = (A+C)x^2+(2A+B)x+2B$

$\Rightarrow A+C = 1; 2A+B = 1; 2B = 1$

$\Rightarrow A = \frac{1}{4}; B = \frac{1}{2}\ ; C = \frac{3}{4}$

$\therefore \int\frac{x^2+x+1}{x^2(x+2)}dx = \frac{1}{4}\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x^2}+\frac{3}{4}\int\frac{dx}{x+2}$

$= \frac{1}{4}logx-\frac{1}{2x}+\frac{3}{4}log(x+2)+C$

Q3. $\int\frac{x^2+1}{(x-1)^2(x+3)}$dx

Sol.$Let\ \frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$

$\Longrightarrow x^2 + 1 = A(x^2 + 2x - 3) + B(x+3) + C(x^2 - 2x + 1)$

$= (A+C)x^2 + (2A + B - 2C)x + (-3A + 3B + C)$

$\Longrightarrow\ A + C = 1\ ;\ 2A + B - 2C = 0\ ;\ -3A + 3B + C = 1$

$\Longrightarrow A = \frac{3}{8}; B = \frac{1}{2}; C = \frac{5}{8}$

$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)} dx = \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{5}{8}\int \frac{dx}{x+3}$

$= \frac{3}{8} log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8} log|x+3| + c$

LEVEL III

Q1. $\int \frac{8}{(x+2)(x^2+4)} dx$

Sol.$let\ \frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{(x^2+4)}$

$8 = A(x^2+4) + (Bx + C)(x+2)$

$= (A+B)x^2 + (2B + C)x + (2C + 4A)$

$\Longrightarrow A + B = 0\ ;\ 2B + C = 0\ ;\ 2C + 4A = 8$

$\Longrightarrow A = 1; B = -1; C = 2$

$\int \frac{8}{(x+2)(x^2+4)} dx = \frac{1}{x+2} + \frac{(-x+2)}{(x^2+4)}$

$= \int \frac{1}{x+2}$dx - $\frac{1}{2}\int \frac{(2x)}{(x^2+4)} dx + 2\int \frac{dx}{(x^2+4)}$

=$log|x+2| - \frac{1}{2} log|(x^2+4)| + \frac{2}{2} tan^{-1}\left(\frac{x}{2}\right) + C$

=$log|x+2| - \frac{1}{2} log|(x^2+4)| + tan^{-1}\left(\frac{x}{2}\right) + C$

Q2. $\int \frac{dx}{sinx + sin2x}$

Sol.$\int \frac{dx}{sinx+sin2x} = \int \frac{dx}{sinx+2sinxcosx} = \int \frac{dx}{sinx(1+2cosx)}$

$= \int \frac{sinxdx}{sin^2x(1+2cosx)} = \int \frac{sinxdx}{(1-cos^2x)(1+2cosx)} = \int \frac{sinxdx}{(1-cosx)(1+cosx)(1+2cosx)}$

$Put\ cosx = t \implies -sinxdx = dt$

$$\implies \int \frac{sinxdx}{(1-cosx)(1+cosx)(1+2cosx)} = \int \frac{-dt}{(1-t)(1+t)(1+2t)}$$

$=\int \frac{dt}{(t+1)(t-1)(1+2t)}$

$$Let \frac{1}{(t+1)(t-1)(1+2t)} = \frac{A}{(t-1)} + \frac{B}{(t+1)} + \frac{C}{(1+2t)}$$

$\therefore 1 = A(t+1)(1+2t) + B(t-1)(1+2t) + C(t-1)(t+1)$

$1 = (2A+2B+C)t^2 + (3A-B)t + (A-B-C)$

$\implies 2A+2B+C = 0;\ 3A-B = 0; A-B-C = 1$

$$\implies A = \frac{1}{6}; B = \frac{1}{2}; C = \frac{-4}{3}$$

$$\int \frac{dt}{(t+1)(t-1)(1+2t)} = \frac{1}{6}\int \frac{dt}{t-1} + \frac{1}{2}\int \frac{dt}{t+1} - \frac{4}{3}\int \frac{dt}{1+2t}$$

$= \frac{1}{6}log|t-1| + \frac{1}{2}log|t+1| - \frac{4}{3}\frac{log|1+2t|}{2}$+C

$= \frac{1}{6}log|t-1| + \frac{1}{2}log|t+1| - \frac{2}{3}log|1+2t|$+C

$= \frac{1}{6}log|cosx-1| + \frac{1}{2}log|cosx+1| - \frac{2}{3}log|1+2cosx|$+C

Q3.$\int \frac{dx}{1+x^3}$

Sol. I=$\int \frac{dx}{1+x^3} = \int \frac{dx}{(1+x)(1-x+x^2)}$

$$let \frac{1}{(1+x)(1-x+x^2)} = \frac{A}{x+1} + \frac{Bx+C}{(1-x+x^2)}$$

$\implies 1 = A(1-x+x^2)+(Bx+C)(x+1)$

1= $(A+B)x^2 + (-A+B+C)x + (A+C)$

$\Longrightarrow\ A+B=0;\ -A+B+C=0;\ A+C=1$

$$\Longrightarrow A=\frac{1}{3}; B=-\frac{1}{3}; C=\frac{2}{3}$$

$$\therefore \int \frac{dx}{(1+x)(1-x+x^2)}=\frac{1}{3}\int \frac{dx}{1+x}-\frac{1}{3}\int \frac{x-2}{1-x+x^2}dx$$

$$=\frac{1}{3}\int \frac{dx}{1+x}-\frac{1}{3}\int \frac{\frac{1}{2}(2x-1)-\frac{3}{2}}{1-x+x^2}dx$$

$$=\frac{1}{3}log(1+x)-\frac{1}{6}\int \frac{2x-1}{1-x+x^2}dx+\frac{1}{2}\int \frac{dx}{1-x+x^2}$$

$$=\frac{1}{3}log(1+x)-\frac{1}{6}log|1-x+x^2|+\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}log(1+x)-\frac{1}{6}log|1-x+x^2|+\frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$=\frac{1}{3}log(1+x)-\frac{1}{6}log|1-x+x^2|+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

(v) Integration by Parts

LEVEL I

Q1. $\int xsec^2xdx$

Sol. $\int xsec^2xdx = xtanx-\int 1.tanxdx$

$$= xtanx+log|secx|+c$$

Q2. $\int logxdx$

Sol. $\int logxdx=\int logx.1\,dx = logx\,.x-\int \frac{1}{x}.xdx$

$$= logx.x-x+c$$

$$= x(logx-1)+c$$

Q3 $\int e^x(tanx+logsecx)dx$

Sol. Since $\int e^x(f(x)+f'(x))dx = e^xf(x)+c$

Here, $f(x)=logsecx$

$\therefore \int e^x(tanx + logsecx)dx = \ e^x logsecx + c$

LEVEL II

Q1. $\int sin^{-1} x\, dx$

Sol.$\int sin^{-1} x\, dx = \int sin^{-1} x.1\, dx = sin^{-1} x.x - \int \frac{1}{\sqrt{1-x^2}}.xdx$

$$= sin^{-1} x.x + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}.(-2x)\, dx$$

$$= sin^{-1} x.x + \frac{1}{2}.2\sqrt{1-x^2} + C$$

$$= x\, sin^{-1} x + \sqrt{1-x^2} + C$$

Q2.$\int x^2\, sin^{-1} x\, dx$

Sol.$\int x^2\, sin^{-1} x\, dx$

Put $sin^{-1} x = \theta \implies x = sin\theta \implies dx = cos\theta d\theta$

$$\therefore \int x^2\, sin^{-1} x\, dx = \int sin^2\theta.\theta.cos\theta d\theta$$

$$= \int \theta.sin^2\theta cos\theta d\theta = \theta.\frac{sin^3\theta}{3} - \int 1.\frac{sin^3\theta}{3}d\theta$$

$$= \ \theta.\frac{sin^3\theta}{3} - \frac{1}{12}\int (3sin\theta - sin3\theta)d\theta$$

$$= \ \theta.\frac{sin^3\theta}{3} - \frac{1}{4}\int sin\theta d\theta - \frac{1}{12}\int sin3\theta\, d\theta$$

$$= \ \theta.\frac{sin^3\theta}{3} + \frac{1}{4}cos\theta - \frac{1}{36}cos3\theta + C$$

$$= \theta.\frac{sin^3\theta}{3} + \frac{1}{4}\sqrt{1-sin^2\theta} - \frac{1}{36}[4cos^3\theta - 3cos\theta] + C$$

$$=\theta.\frac{sin^3\theta}{3} + \frac{1}{4}\sqrt{1-sin^2\theta} - \frac{1}{9}(1-sin^2\theta)^{\frac{3}{2}} + \frac{1}{12}\sqrt{1-sin^2\theta} + C$$

$$= \ \theta.\frac{sin^3\theta}{3} + \frac{1}{3}\sqrt{1-sin^2\theta} - \frac{1}{9}(1-sin^2\theta)^{\frac{3}{2}} + C$$

$= \frac{x^3}{3} sin^{-1} x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + C$

Q3 $\int \frac{x\, sin^{-1} x}{\sqrt{1-x^2}} dx$

Sol. $put\ sin^{-1} x = \theta \implies x = sin\theta \implies dx = cos\theta d\theta$

$\therefore \int \frac{x\, sin^{-1} x}{\sqrt{1-x^2}} dx = \int \frac{\theta sin\theta . cos\theta d\theta}{cos\theta} = \int \theta sin\theta d\theta$

$= \theta(-cos\theta) - \int 1.(-cos\theta) d\theta$

$= -\theta cos\theta + sin\theta + C$

$= -\sqrt{1-x^2}\, sin^{-1} x + x + C$

Q4 $\int cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Sol. $putx = tan\theta\ \therefore dx = sec^2\theta d\theta$

$\therefore \int cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right) sec^2 d\theta = \int cos^{-1} cos2\theta\, sec^2\theta d\theta$

$= 2\int \theta\, sec^2\theta d\theta = 2\{\theta tan\theta - logsec\theta\} + C$

$= 2x\, tan^{-1} x - log|1 + x^2| + C$

Q5. $\int sec^3$xdx

Sol. I= $\int sec^3$xdx= $\int secx(sec^2x)dx$

$= secxtanx - \int secxtanx(tanx)\, dx$

$= secxtanx - \int secxtan^2x\, dx$

$= secxtanx - \int secx(sec^2x - 1)dx$

$= secxtanx - \int sec^3$x dx+ $\int secxdx$

$\therefore I = secxtanx - I + log\left|tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

2I= $secxtanx + log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

I=$\frac{1}{2}\left[secxtanx + log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right] + C$

LEVEL III

Q1.$\int cos(logx)\, dx$

Sol.$putlogx = t \implies x = e^t \implies dx = e^t dt$

I=$\int coste^t dt = e^t sint - \int e^t sintdt$

$$= e^t sint - [e^t(-cost) - \int e^t(-cost)dt]$$

$$I = e^t sint + e^t cost - I$$

$$\therefore 2I = e^t sint + e^t cost$$

$$\implies I = \frac{1}{2}[e^t sint + e^t cost] + C$$

Q2 $\int e^x \frac{(1+x)}{(2+x)^2} dx$

Sol.$\int e^x \frac{(1+x)}{(2+x)^2} dx = \int e^x \frac{(2+x-1)}{(2+x)^2} dx$=$\int e^x \left(\frac{1}{2+x} - \frac{1}{(2+x)^2}\right) dx$

$= \frac{e^x}{2+x} + C$ $\quad [\therefore \int e^x(f(x) + f'(x))dx = e^x f(x) + c]$

Q3$\int \frac{logx}{(1+logx)^2} dx$

Sol. $\int \frac{logx}{(1+logx)^2} dx = \int \frac{(1+logx-1)}{(1+logx)^2} dx = \int \left[\frac{1}{1+logx} - \frac{1}{(1+logx)^2}\right] dx$

=$\int \left[\frac{dx}{1+logx} - \frac{dx}{(1+logx)^2}\right]$

$$= \frac{1}{1+logx}.x - \int \frac{-1}{(1+logx)^2}.\frac{1}{x}.xdx - \int \frac{dx}{(1+logx)^2}$$

=$\frac{x}{1+logx} + \int \frac{dx}{(1+logx)^2} - \int \frac{dx}{(1+logx)^2} + c = \frac{x}{1+logx} + c$

Q4. $\int \left(\frac{2+sin2x}{1+cos2x}\right) e^x\, dx$

Sol.$\int \left(\frac{2+sin2x}{1+cos2x}\right) e^x\, dx = \int \left[\frac{e^x.2}{2cos^2x} + \frac{e^x 2sinxcosx}{2cos^2x}\right] dx$

$$= \int e^x sec^2 xdx + \int e^x tanxdx$$

$Integrate first by parts,$

$I = e^x tanx - \int e^x tanxdx + \int e^x tanxdx$

$$= e^x tanx + C$$

Q5$\int e^{2x} cos3xdx$

Sol. Let I = $\int e^{2x} cos3xdx$

$= cos3x \frac{e^{2x}}{2} - \int (-3sin3x) \frac{e^{2x}}{2} dx$

$= \ cos3x \frac{e^{2x}}{2} + \frac{3}{2} \int e^{2x}\, sin3xdx$

$= \ cos3x \frac{e^{2x}}{2} + \frac{3}{2}$(sin3x $\frac{e^{2x}}{2} - \int 3cos3x \frac{e^{2x}}{2} dx)$

$= \ cos3x \frac{e^{2x}}{2} + \frac{3}{4} sin3x e^{2x} - \frac{9}{4}$I

$I + \frac{9}{4}$I=$cos3x \frac{e^{2x}}{2} + \frac{3}{4} sin3x e^{2x}$

$\frac{13}{4} I = cos3x \frac{e^{2x}}{2} + \frac{3}{4} sin3x e^{2x}$

$I = \frac{4}{13}\left(cos3x \frac{e^{2x}}{2} + \frac{3}{4} sin3x e^{2x}\right) + C$

$= \frac{1}{13}(2cos3x + 3sin3x)e^{2x} + C$

(vi) Some Special Integrals

LEVEL I

Q1.$\int \sqrt{4 + x^2}\, dx$

Sol.$\int \sqrt{4 + x^2}\, dx = \frac{x}{2}\sqrt{4 + x^2} + \frac{2^2}{2} log\left|x + \sqrt{4 + x^2}\right| + C$

Q2. $\int \sqrt{1-4x^2}$dx

Sol. $2\int \sqrt{\left(\frac{1}{2}\right)^2 - x^2}\, dx$

$=2[\frac{x}{2}\sqrt{\left(\frac{1}{2}\right)^2 - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} sin^{-1}\left(\frac{x}{1/2}\right)] + C$

$$= x\sqrt{\left(\frac{1}{2}\right)^2 - x^2} + \frac{1}{4} sin^{-1} 2x + C$$

LEVEL II

Q1. $\int \sqrt{x^2+4x+6}\, dx$

Sol. $\therefore x^2 + 4x + 6 = x^2 + 2.2. x + 4 - 4 + 6$

$= (x+2)^2 + \left(\sqrt{2}\right)^2$

$\therefore \int \sqrt{x^2+4x+6}\, dx = \int \sqrt{(x+2)^2 + \left(\sqrt{2}\right)^2}$dx

$$= \frac{(x+2)}{2}\sqrt{x^2+4x+6} + \frac{\left(\sqrt{2}\right)^2}{2} log\left|(x+2) + \sqrt{x^2+4x+6}\right| + C$$

$$= \frac{(x+2)}{2}\sqrt{x^2+4x+6} + log\left|(x+2) + \sqrt{x^2+4x+6}\right| + C$$

Q2. $\int \sqrt{1-4x-x^2}\, dx$

Sol. $\therefore 1 - 4x - x^2 = \left(\sqrt{5}\right)^2 - (x+2)^2$

$.\therefore \int \sqrt{1-4x-x^2}\, dx = \int \sqrt{\left(\sqrt{5}\right)^2 - (x+2)^2}\, dx$

$$= \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2} sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

LEVEL III

Q1. $\int (x+1)\sqrt{1-x-x^2}\, dx$

Sol. Let $x + 1 = A\frac{d}{dx}(1 - x - x^2) + B$

$x + 1 = A(-1 - 2x) + B$

$= \ -2Ax + (B - A)$

$\Longrightarrow A = -\dfrac{1}{2}; B = \dfrac{1}{2}$

$\therefore \displaystyle\int (x+1)\sqrt{1 - x - x^2}\, dx$

$= -\frac{1}{2}\int (- 1 - 2x)\ \sqrt{1 - x - x^2}\text{dx} + \frac{1}{2}\int \sqrt{1 - x - x^2}dx$

$= -\dfrac{1}{2}\dfrac{(1 - x - x^2)^{3/2}}{3/2} + \dfrac{1}{2}\displaystyle\int \sqrt{\left(\dfrac{\sqrt{5}}{2}\right)^2 - \left(x + \dfrac{1}{2}\right)^2}\, dx$

$= -\frac{1}{3}(1 - x - x^2)^{3/2} + \frac{1}{2}\{\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2} + \frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2} sin^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}} + C$

$== -\dfrac{1}{3}(1 - x - x^2)^{3/2} + \dfrac{1}{4}\left(x + \dfrac{1}{2}\right)\sqrt{1 - x - x^2} + \dfrac{5}{8} sin^{-1}\left(\dfrac{2x+1}{\sqrt{5}}\right) + C$

Q2.$\int (x-5)\sqrt{x^2 + x}\, dx$

Sol. Let $(x - 5) = A\frac{d}{dx}(x^2 + x) + B$

$x - 5 = A(2x + 1) + B$

$\Longrightarrow\ x - 5 = 2Ax + (A + B)$

$\Longrightarrow A = \dfrac{1}{2}; B = -\dfrac{11}{2}$

$\therefore \displaystyle\int (x-5)\sqrt{x^2 + x}\, dx = \dfrac{1}{2}\int (2x+1)\ \sqrt{x^2 + x}dx - \dfrac{11}{2}\int \sqrt{x^2 + x}dx$

$= \dfrac{1}{2}\dfrac{(x^2 + x)^{3/2}}{3/2} - \dfrac{11}{2}\displaystyle\int \sqrt{(x + \dfrac{1}{2})^2 - \left(\dfrac{1}{2}\right)^2}\, dx$

$= \dfrac{1}{2}\dfrac{(x^2 + x)^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{11}{2}\left\{\left(\dfrac{x + \frac{1}{2}}{2}\right)\sqrt{x^2 + x} - \dfrac{\left(\frac{1}{2}\right)^2}{2} log\left|x + \dfrac{1}{2} + \sqrt{x^2 + x}\right|\right\} + C$

$$= \frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{11}{2}\left\{\left(\frac{x+\frac{1}{2}}{2}\right)\sqrt{x^2+x} - \frac{1}{8}log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|\right\} + C$$

Definite Integrals

(i) Definite Integrals based upon types of indefinite integrals

LEVEL-I

Q1. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

Sol. $\int_0^1 \frac{2x+3}{5x^2+1}dx = 2\int \frac{x}{5x^2+1} + 3\int \frac{1}{5x^2+1}dx$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1} + \frac{3}{5}\int \frac{1}{x^2+\left(\frac{1}{\sqrt{5}}\right)^2}dx$$

$$= \left[\frac{1}{5}log(5x^2+1) + \frac{3}{\sqrt{5}}tan^{-1}\left(\sqrt{5}x\right)\right]_0^1$$

$$= \frac{1}{5}(log6 - log1) + \frac{3}{\sqrt{5}}(tan^{-1}\sqrt{5} - tan^{-1}0)$$

$$= \frac{1}{5}\text{log}6 + \frac{3}{\sqrt{5}}tan^{-1}\sqrt{5}$$

Q2. $\int_0^{\frac{\pi}{2}} \sqrt{sinx}\, cos^5 x dx$

Sol. $\int_0^{\frac{\pi}{2}} \sqrt{sinx}\,(1-sin^2 x)^2 cosxdx$

$Put sinx = t;\ cosxdx = dt,$

$When x = 0 \Rightarrow t = 0\ \&x = \frac{\pi}{2} \Rightarrow t = 1$

$$= \int_0^1 \sqrt{t}\,(1-t^2)^2 dt$$

$$= \int_0^1 (t^{1/2} + t^{\frac{9}{2}} - 2t^{5/2})dt = \left[\frac{t^{3/2}}{3/2} + \frac{t^{11/2}}{11/2} - 2\frac{t^{7/2}}{7/2}\right]_0^1$$

$$= \left(\frac{2}{3} + \frac{2}{11} - \frac{4}{7}\right) = \frac{64}{231}$$

Q3. $\int_0^2 x\sqrt{x+2}\,dx$

Sol. Put $x + 2 = t \Rightarrow dx = dt$

$whenx = 0 \implies t = 2\,; x = 2 \implies t = 4$

$\implies \int_2^4 (t^{3/2} - 2t^{1/2})dt = \left[\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{3/2}\right]_2^4$=$[\left(\frac{64}{5} - \frac{32}{3}\right) - (\frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3})]$

$= \frac{32}{15} + \frac{16}{15}\sqrt{2} = \frac{16\sqrt{2}}{15}(\sqrt{2} + 1)$

LEVEL-II

Q1.$\int_1^2 \frac{5x^2}{x^2+4x+3}$dx

Sol.$\int_1^2 \frac{5x^2}{x^2+4x+3}dx = 5\int_1^2 \frac{x^2}{x^2+4x+3}dx = 5\int_1^2 (1 - \frac{4x+3}{x^2+4x+3})dx = 5\int_1^2 (1 - \frac{2(2x+4)-5}{x^2+4x+3})\,dx$

$$= 5\int_1^2 dx - 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx + 25\int_1^2 \frac{1}{x^2+4x+3}dx$$

$$= 5\int_1^2 dx - 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx + 25\int_1^2 \frac{1}{(x+2)^2-1}dx$$

$$= \left[5x - 10\,log|x^2 + 4x + 3| + \frac{25}{2}log\left|\frac{x+2-1}{x+2+1}\right|\right]_1^2$$

$= 5 - 10$(log15-log8)+$\frac{25}{2}$(log$\frac{3}{5} - log\,\frac{1}{2}$)

$$= 5 - 10log\frac{15}{8} + \frac{25}{2}log\frac{6}{5}$$

Q2.$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}dx$

Sol.$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}dx = \int_1^2 \frac{e^{2x}}{x}dx - \int_1^2 \frac{1}{2x^2}e^{2x}dx$

Integrate first by parts,

$$= \left[\frac{e^{2x}}{2x}\right]_1^2 + \int_1^2 \frac{1}{2x^2}e^{2x} - \int_1^2 \frac{1}{2x^2}e^{2x}$$

$$= \frac{e^2}{4}(e^2 - 2)$$

(ii) Definite integrals as a limit of sum

LEVEL I

Q1. $\int_0^2 (x + 2)dx$

Sol. We divide the interval [0,2] in n-equal sub interval each of length h.

$$\therefore nh = 2 - 0 = 2$$

Here, $f(x) = x + 2$

$$f(0) = 0 + 2 = 2$$
$$f(0+h) = f(h) = h + 2$$
$$f(0+2h) = f(2h) = 2h + 2$$
$$f(0+3h) = f(3h) = 3h + 2$$

……………………………………………………………………………

$$f(0 + (n-1)h) = f\big((n-1)h\big) = (n-1)h + 2$$

$\therefore$ By the definition of definite integrals as the limit of sum,

$$\int_0^2 (x+2)dx$$
$$= \lim_{h\to 0} h\{2 + (h+2) + (2h+2) + (3h+2) + \ldots\ldots. + ((n-1)h + 2)\}$$
$$= \lim_{h\to 0} h\{(2 + 2 + \cdots n\ times) + h[1 + 2 + \cdots + (n-1)]\}$$
$$= \lim_{h\to 0} h\{2n + h\frac{n(n-1)}{2}\} = \lim_{h\to 0}\{2nh + \frac{nh(nh-h)}{2}\}$$
$$= \lim_{h\to 0}\{2.2 + \frac{2.(2-h)}{2}\} = 4 + \frac{2.(2-0)}{2} = 4 + 2 = 6$$

Q2 $\int_0^4 (1+x)dx$

Sol. We divide the interval [0,4]in $\mathrm{n} -$ equal subinterval each of length h.

$$\therefore nh = 4 - 0 = 4$$

Here, $f(x) = x + 1$

$$f(0) = 0 + 1 = 1$$
$$f(0+h) = f(h) = h + 1$$
$$f(0+2h) = f(2h) = 2h + 1$$
$$f(0+3h) = f(3h) = 3h + 1$$

……………………………………………………………………………

$$f(0 + (n-1)h) = f\big((n-1)h\big) = (n-1)h + 1$$

$\therefore$ By the definition of definite integrals as the limit of sum,

$$\int_0^4 (1+x)dx = \lim_{h\to 0} h\{1 + (h+1) + (2h+1) + (3h+1) + \ldots\ldots. + ((n-1)h + 1)\}$$
$$= \lim_{h\to 0} h\{(1 + 1 + \cdots + 1n\ times) + h[1 + 2 + \cdots + (n-1)]\}$$
$$= \lim_{h\to 0} h\{n + h\frac{n(n-1)}{2}\} = \lim_{h\to 0}\{nh + \frac{nh(nh-h)}{2}\}$$
$$= \lim_{h\to 0}\{4 + \frac{4.(4-h)}{2}\} = 4 + \frac{4.(4-0)}{2} = 4 + 8 = 12$$

LEVEL II

Q1. $\int_1^2 (3x^2 - 1)\, dx$

Sol. .We divide the interval [1,2]in n − equal sub interval each of ength h.

$$\therefore nh = 2 - 1 = 1$$

Here, $f(x) = 3x^2 - 1$

$$f(1) = 3(1)^2 - 1 = 2$$

$$f(1+h) = 3(1+h)^2 - 1 = 3(2h + h^2) + 2$$

$$f(1+2h) = 3(1+2h)^2 - 1 = 3(2.2h + 2^2h^2) + 2$$

……………………………………………………………………………

$$f(1+(n-1)h) = 3(1+(n-1)h)^2 - 1 = 3\{2.(n-1)h + (n-1)^2h^2\} + 2$$

∴ By the definition of definite integrals as the limit of sum,

$$\int_1^2 (3x^2 - 1)dx$$

$= lim_{h\to 0}\, h\{2 + [3(2h + h^2) + 2] + [3(2.2h + 2^2h^2) + 2] \;\; \dots \dots . +3[2.(n-1)h +$
$(n-1)^2h^2]\}$+2

$= \lim_{h\to 0} h\{(2 + 2 + \cdots + n\ times) + 6h[1 + 2 + \cdots + (n-1)] + 3h^2\{1^2 + 2^2 + \cdots \dots$
$+ (n-1)^2\}$

$= \lim_{h\to 0} h\{2n + 6h\dfrac{n(n-1)}{2} + 3h^2\dfrac{n(n-1)(2n-1)}{6}\}$

$$= \lim_{h\to 0}\{2nh + \frac{6nh(nh-h)}{2} + 3\frac{nh(nh-h)(2nh-h)}{6}\}$$

$= \lim_{h\to 0}\{2 + \dfrac{6.(1-h)}{2} + 3\dfrac{1.(1-h)(2-h)}{6}\} = 2 + 3 + 1 = 6$

Q2. $\int_0^3 (x^2 + 1)dx$

Sol.We divide the interval [0,3] in n − equal subinterval each of length h.

$$\therefore nh = 3 - 0 = 3$$

Here, $f(x) = x^2 + 1$

$$f(0) = (0)^2 + 1 = 1$$

$$f(0+h) = f(h) = h^2 + 1$$

$$f(0+2h) = f(2h) = (2h)^2 + 1$$

……………………………………………………………………………

$$f(0+(n-1)h) = f\big((n-1)h\big) = ((n-1)h)^2 + 1$$

∴ By the definition of definite integrals as the limit of sum,

$$\int_0^3 (x^2 + 1)dx = \lim_{h\to 0} h\{1 + [h^2 + 1] + [(2h)^2 + 1] \;\; \dots \dots . +[((n-1)h)^2 + 1]\}$$

$= \lim_{h\to 0} h\{(1 + 1 + \cdots + n\ times) + h^2\{1^2 + 2^2 + \cdots \dots + (n-1)^2\}$

$= \lim_{h\to 0} h\{n + h^2 \frac{n(n-1)(2n-1)}{6}\} = \lim_{h\to 0}\{nh + \frac{nh(nh-h)(2nh-h)}{6}\}$

$= \lim_{h\to 0}\{3 + \frac{3.(3-h)(6-h)}{6}\} = 3 + 9 = 12$

(iii) Properties of definite Integrals

LEVEL I

Q1. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx$

Sol. $Let I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{tan\, x}}{1+\sqrt{tan\, x}} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\frac{\sqrt{sin\, x}}{\sqrt{cos\, x}}}{1+\frac{\sqrt{sin\, x}}{\sqrt{cos\, x}}} dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\, x}}{\sqrt{cos\, x}+\sqrt{sin\, x}} dx$……….(1)

I=$\int_0^{\frac{\pi}{2}} \frac{\sqrt{sin(\frac{\pi}{2}-x)}}{\sqrt{cos(\frac{\pi}{2}-x)}+\sqrt{sin(\frac{\pi}{2}-x)}}$dx $= \int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}}{\sqrt{sin\, x}+cosx} dx$……..(2)

Adding (1) & (2) we get,

2I=$\int_0^{\frac{\pi}{2}} \frac{\sqrt{cosx}+\sqrt{sin\, x}}{\sqrt{sin\, x}+cosx} dx = \int_0^{\frac{\pi}{2}} dx = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

Q2.$\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}} dx$

Sol.Let I=$\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}} dx$…………(1)

$= \int_1^3 \frac{\sqrt{4-(4-x)}}{\sqrt{4-x}+\sqrt{4-(4-x)}} dx = \int_1^3 \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}} dx$…………..(2)

Adding (1) & (2) we get,

2I=$\int_1^3 \frac{\sqrt{4-x}+\sqrt{x}}{\sqrt{x}+\sqrt{4-x}} dx = \int_1^3 dx = [x]_1^3 = 3 - 1 = 2$

$\therefore I = 1$

Q3 .$\int_0^{\frac{\pi}{2}} \frac{sin^4 x}{sin^4 x+cos^4 x} dx$

Sol. Let I=$\int_0^{\frac{\pi}{2}} \frac{sin^4 x}{sin^4 x+cos^4 x} dx$……….(1)

$= \int_0^{\frac{\pi}{2}} \frac{sin^4(\frac{\pi}{2}-x)}{sin^4(\frac{\pi}{2}-x)+cos^4(\frac{\pi}{2}-x)} dx = \int_0^{\frac{\pi}{2}} \frac{cos^4 x}{cos^4 x+sin^4 x} dx$……….(2)

Adding (1) & (2) we get,

$2I = \int_0^{\frac{\pi}{2}} dx = \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

LEVEL II

Q1. $\int_0^{\frac{\pi}{2}} \frac{x}{sin\, x+cos\, x} dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{x}{sin\, x+cos\, x} dx$……………(1)

$$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2} - x)}{sin(\frac{\pi}{2} - x) + cos(\frac{\pi}{2} - x)} dx$$

$= \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2}-x)}{cosx+sinx} dx$……….(2)

Adding (1) & (2) we get,

$2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{dx}{cosx+sinx} = \frac{\pi}{2\sqrt{2}} \int_{0}^{\frac{\pi}{2}} \frac{dx}{\frac{1}{\sqrt{2}}cosx+\frac{1}{\sqrt{2}}sinx} = \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{sin\frac{\pi}{4}cosx+cos\frac{\pi}{4}sinx}$

$= \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{sin\frac{\pi}{4}cosx+cos\frac{\pi}{4}sinx} = \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{dx}{Sin(\frac{\pi}{4}+x)}$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\frac{\pi}{2}} cosec(\frac{\pi}{4} + x)dx = \left[\frac{\pi}{2\sqrt{2}} log \left|cosec\left(\frac{\pi}{4} + x\right) - cot(\frac{\pi}{4} + x)\right|\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2\sqrt{2}} log\left|\sqrt{2} + 1\right| \qquad \int$$

Q2. $= \int_0^{\pi} \frac{xsinx}{1+cos^2 x} dx$ ……………..(i)

$$= \int_0^{\pi} \frac{(\pi - x)\, sin(\pi - x)\, dx}{1 + cos^2(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\, sin\, x\, dx}{1 + cos^2 x} \ldots\ldots\ldots (ii)$$

$Adding (i)\&(ii), weget$

$$2I = \int_0^{\pi} \frac{\pi\, sin\, x\, dx}{1 + cos^2 x}$$

$$I = \frac{1}{2}\int_0^{\pi} \frac{\pi\, sin\, x\, dx}{1 + cos^2 x} Putcosx = t \Longrightarrow -sinxdx = dt$$

$$Whenx \to 0; t \to 1\ \&x \to \pi; t \to -1$$

$$I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1 + t^2} = -\frac{\pi}{2}[tan^{-1} t]_1^{-1}$$

$$I = -\frac{\pi}{2}[tan^{-1}(-1) - tan^{-1} 1] = -\frac{\pi}{2}\left[\frac{-\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}$$

Q3. $\int_0^{\pi} \frac{xtanx}{secxcosecx} dx$

Sol. Let I= $\int_0^{\pi} \frac{xtanx}{secxcosecx} dx$…………………(1)

$= \int_0^{\pi} \frac{(\pi - x)tan(\pi - x)}{sec(\pi - x)\ cosec(\pi - x)} dx$

$= \int_0^{\pi} \frac{(\pi - x)(-tanx)}{-secxcosecx} dx$

$= \int_0^{\pi} \frac{(\pi - x)(tanx)}{secxcosecx} dx$……………..(2)

Adding (1) & (2) we get,

2I= $\pi \int_0^{\pi} \frac{tanxdx}{secxcosecx} = \pi \int_0^{\pi} sin^2 x\, dx = \pi \int_0^{\pi} \frac{1 - cos2x}{2} dx$

$= \frac{\pi}{2}\int_0^{\pi} dx - \frac{\pi}{2}\int_0^{\pi} cos2xdx$

$= \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(sin2\pi - sin0)$

$= \frac{\pi^2}{2}$

$\therefore I = \frac{\pi^2}{4}$

Q4. $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tanx}}$

Sol. Let I= $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tanx}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{cosx}}{\sqrt{cos\,x}+\sqrt{sin\,x}} dx$……………..(1)

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{cos\,(\frac{\pi}{3}+\frac{\pi}{6}-x)}}{\sqrt{cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{sin\left(\frac{\pi}{3}+\frac{\pi}{6}-\right)x}} dx$$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{sinx}}{\sqrt{cos\,x}+\sqrt{sin\,x}} dx$………….(2)

Adding (1)& (2)we get,

2I= $\int_{\pi/6}^{\pi/3} dx = \left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\Rightarrow I = \frac{\pi}{12}$

LEVEL III

Q2. $I = \int_0^{\frac{\pi}{2}} log(sinx)dx \ldots\ldots\ldots\ldots\ldots (i)$

$$I = \int_0^{\frac{\pi}{2}} log\left(sin\left(\frac{\pi}{2}-x\right)\right)dx = \int_0^{\frac{\pi}{2}} logcosxdx \ldots\ldots\ldots\ldots (ii)$$

Adding (i) & (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} [\log sinx + logcosx]dx$$

$$= \int_0^{\frac{\pi}{2}} \log[(sinx).(cosx)]\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{2sinxcosx}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \log 2 . \frac{\pi}{2}$$

Now, $\int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$ Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$$\text{when} x \to 0; t \to 0 \& x \to \frac{\pi}{2}; t \to \pi$$

$$= \int_0^{\pi} \log \sin t \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin(\pi - t) \, dt$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt = \frac{1}{2} . 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt = I$$

$$2I = I - \log 2 \frac{\pi}{2}$$

$$I = \frac{\pi}{2} \log 2$$

Q3.$\int_0^{\pi/4} log(1+tanx)\,dx$

Sol.I=$\int_0^{\pi/4} log(1+tanx)\,dx$

$=\int_0^{\pi/4} log\left(1+tan\,(\frac{\pi}{4}-x)\right)dx$=$\int_0^{\pi/4} log\left|1+\frac{1-tanx}{1+tanx}\right|$dx

$$=\int_0^{\pi/4} log\frac{2}{1+tanx}dx = log2\int_0^{\pi/4} dx - \int_0^{\pi/4} log(1+tanx)\,dx$$

$$= log2\int_0^{\pi/4} dx - I$$

2I=$log2\int_0^{\pi/4} dx$ =log2($\frac{\pi}{4}-0$)=$\frac{\pi}{4}log2$

$$\therefore I=\frac{\pi}{8}log2$$

(iv) Integration of modulus function

LEVEL III

Q1.$\int_2^5[|x-2|+|x-3|+|x-4|]dx$

Sol. $=\int_2^3[|x-2|+|x-3|+|x-4|]dx+\int_3^4[|x-2|+|x-3|+|x-4|]dx+\int_4^5[|x-2|+$ x−3+x−4]dx

$$=\int_2^3[x-2-(x-3)-(x-4)]dx$$

$$+\int_3^4[x-2+x-3-(x-4)]dx+\int_4^5[x-2+x-3+(x-4)]dx$$

$$=\int_2^3(5-x)dx+\int_3^4(x-1)dx+\int_4^5(3x-9)dx$$

$$=\left[5x-\frac{x^2}{2}\right]_2^3+\left[\frac{x^2}{2}-x\right]_3^4+\left[\frac{3x^2}{2}-9x\right]_4^5$$

=$\left[\left(15-\frac{9}{4}\right)-(10-2)\right]+\left[(8-4)-\left(\frac{9}{2}-3\right)\right]+[\left(\frac{75}{2}-45\right)-(24-36)]$

$=\frac{19}{2}$

Q2. $\int_{-1}^{2}|x^3 - x|\, dx$

Sol. W$e\ note\ that\ x^3 - x \geq 0\ on[-1, 0]\ and\ x^3 - x \leq 0\ on\ [0, 1]\ and$

$that\ x^3 - x \geq 0\ on\ [1, 2]$

$$\int_{-1}^{2}|x^3 - x|\, dx = \int_{-1}^{0}(x^3 - x)dx + \int_{0}^{1} -(x^3 - x)dx + \int_{1}^{2}(x^3 - x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{2} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{11}{4}$$

Q3. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[sin|x| - cos|x|]dx$

Sol. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[sin|x| - cos|x|]dx = 2\int_{0}^{\frac{\pi}{2}}[sinx - cosx]dx$

$= 2[-cosx - sinx]_{0}^{\frac{\pi}{2}} = 2(1 + 1) = 4$

TOPIC-7

APPLICATION OF INTEGRATION

(i) Area under Simple Curves

LEVEL I

Q1. Sketch the region of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and find its area, using Integration

Sol.

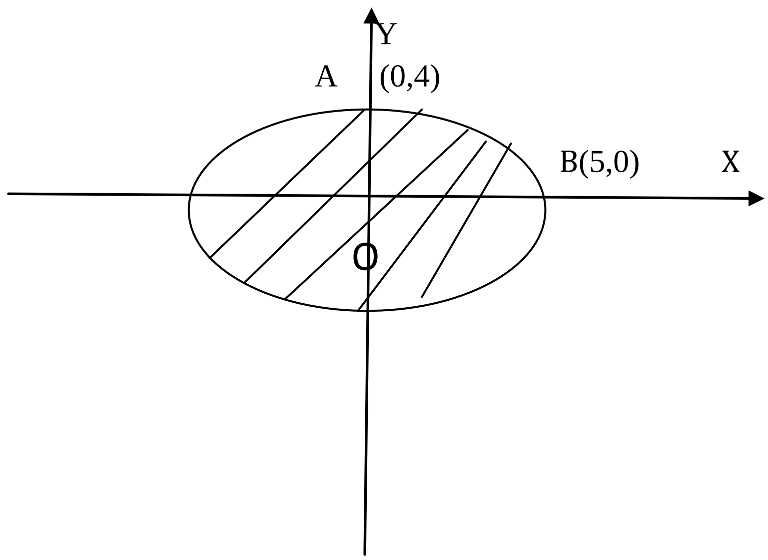


Figure shows the required area of the region of ellipse

Req. area = 4 area of OAB

$$= 4\int_0^5 \frac{4}{5}\sqrt{25-x^2}\,dx$$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}sin^{-1}\frac{x}{5}\right]_0^5$$

$$= \frac{16}{5}[0+\frac{25}{2}sin^{-1}1] = \frac{16}{5}.\frac{25}{2}.\frac{\pi}{2} = 20\pi sq.units$$

Q2. Sketch the region {(x, y) : $4x^2 + 9y^2 = 36$} and find its area, using integration.

Sol. we have, $4x^2 + 9y^2 = 36 \Longrightarrow \frac{x^2}{9}+\frac{y^2}{4} = 1$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

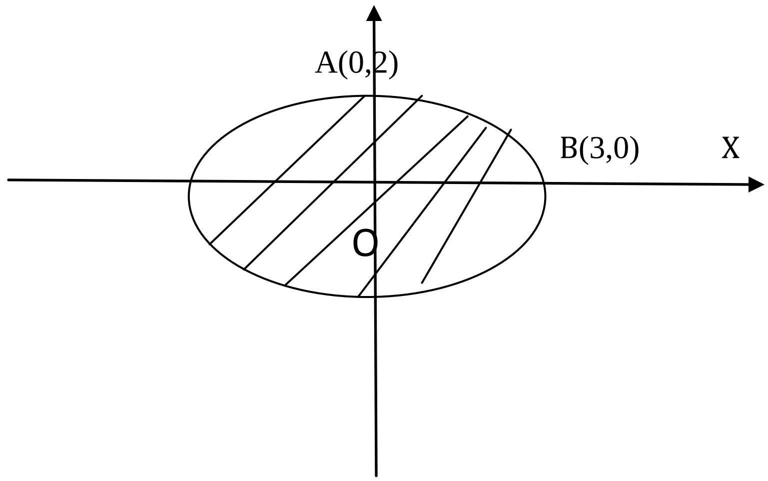


Figure shows the required area of the region of ellipse

Req. area = 4 area of OAB

$$= 4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx = 4\int_0^3 \frac{2}{3}\sqrt{3^2-x^2}\,dx$$

$$= \frac{8}{3}\left[\frac{x}{2}\sqrt{3^2-x^2} + \frac{9}{2}sin^{-1}\frac{x}{3}\right]_0^3$$

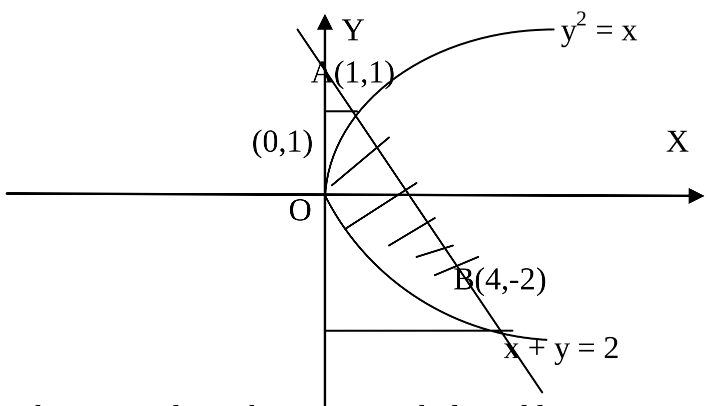
$= \frac{8}{3}[0 + \frac{9}{2}sin^{-1} 1] = \frac{8}{3}.\frac{9}{2}.\frac{\pi}{2} = 6\pi sq.units$

(ii) Area of the region enclosed between Parabola and line

LEVEL II

Q1. Find the area of the region included between the parabola $y^2 = x$ and the line x + y = 2.

Sol.



(0,-2)

Figure shows the required area between parabola and line

Req. area = area of shaded region OAB

$$= \int_{-2}^{1} (X_{Line-} X_{parabola)}\, dy$$

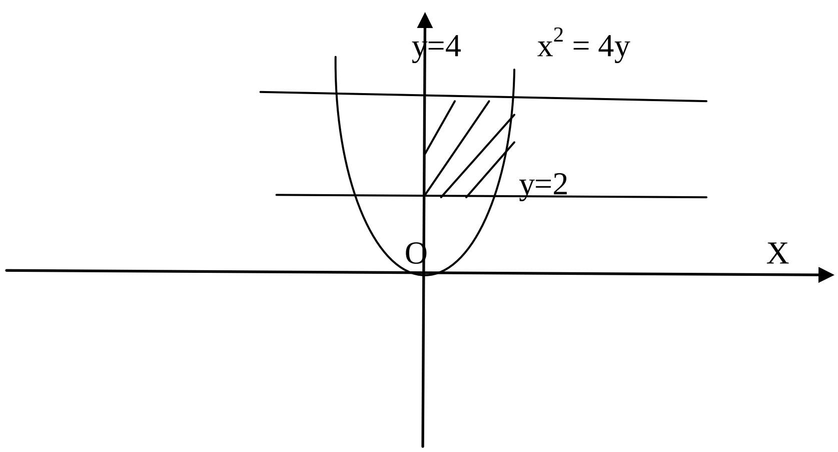
$$= \int_{-2}^{1} (2 - y - y^2) dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1} = (2.1 - \frac{1}{2} - \frac{1}{3}) - (-4 - 2 + \frac{8}{3})$$

$$= \frac{9}{2} sq.units$$

Q2.Find the area of the region bounded by $x^2 = 4y$, y = 2, y = 4 and the y-axis in the first quadrant.

Sol.



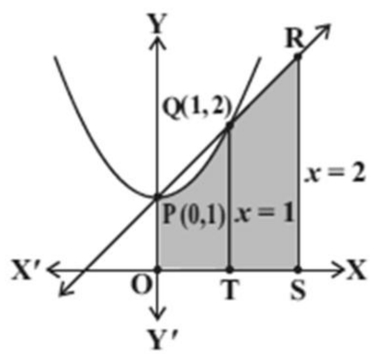
The required area is shown in the figure

Req. area $=\int_2^4 xdy = \int_2^4 2\sqrt{y}dy = 2\left[\frac{y^{3/2}}{3/2}\right]_2^4 = \frac{4}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \frac{32-8\sqrt{2}}{3}$ Sq. units

LEVEL III

Q 1. Solution Let us first sketch the region whose area is to be found out. This region is the intersection of the following regions.

$$A_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$
$$A_2 = \{(x, y) : 0 \le y \le x + 1\}$$
and $$A_3 = \{(x, y) : 0 \le x \le 2\}$$



The points of intersection of $y = x^2 + 1$ and $y = x + 1$ are points P(0, 1) and Q(1, 2). From the Fig , the required region is the shaded region OPQRSTO whose area

= area of the region OTQPO + area of the region TSRQT

$$= \int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx$$

$$= \left[\left(\frac{x^3}{3}+x\right)\right]_0^1 + \left[\left(\frac{x^2}{2}+x\right)\right]_1^2$$

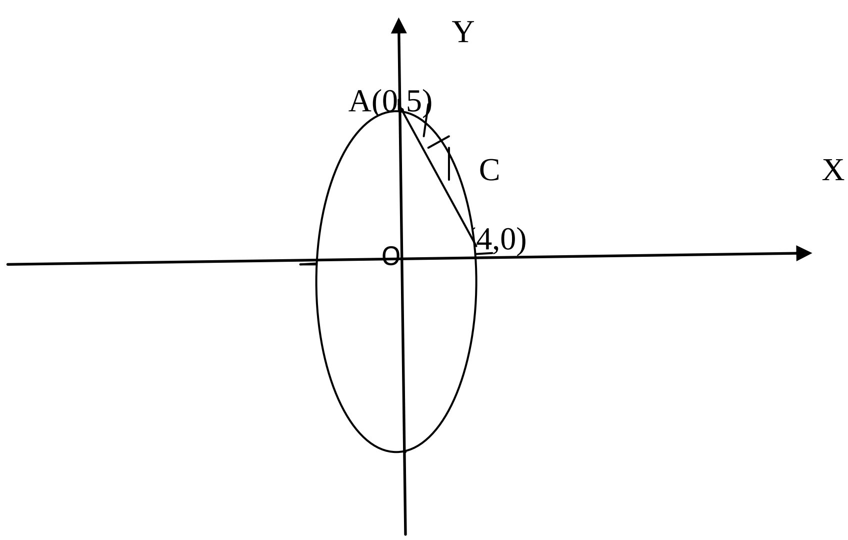
$$= \left[\left(\frac{1}{3}+1\right)-0\right] + \left[(2+2)-\left(\frac{1}{2}+1\right)\right] = \frac{23}{6}$$

(iii) Area of the region enclosed between Ellipse and line

LEVEL II

Q1.Find the area of smaller region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ and the straight line $\frac{x}{4}+\frac{y}{5}=1$.

Sol.



Req.area is shown in the figure = area of ABCA

$$= \int_0^4 [\frac{5}{4}\sqrt{16-x^2} - \frac{5}{4}(4-x)]dx$$

$$= \frac{5}{4}\int_0^4 [\sqrt{16 - x^2} - 4 + x]dx$$

$$= \frac{5}{4}\left[\{\frac{x}{2}\sqrt{16 - x^2} + 8\, sin^{-1}\frac{x}{4} - 4x + \frac{x^2}{2}\}\right]_0^4$$

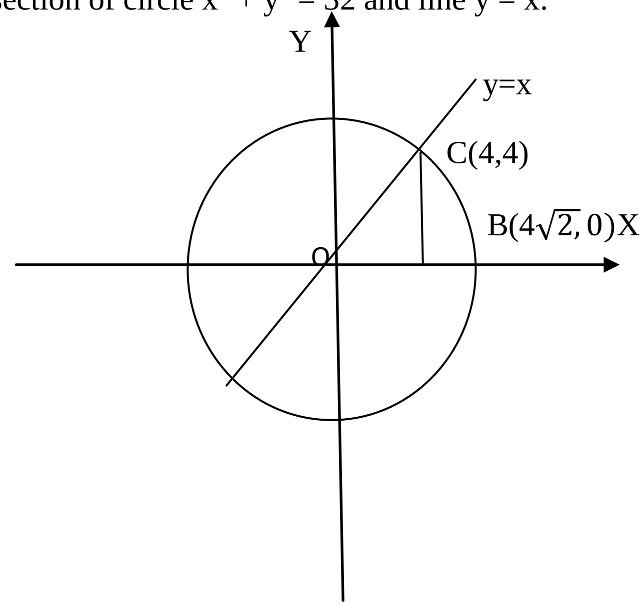
$$= \frac{5}{4}\left\{0 + 8.\frac{\pi}{2} - 4.4 + 8\right\} = \frac{5}{4}\{4\pi - 8\} = 5(\pi - 2)Sq.units$$

(iv) Area of the region enclosed between Circle and line

Level II

Q1. Find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$

Sol. The point of intersection of circle $x^2 + y^2 = 32$ and line y = x.
is (4,4)



Req Area = area of OBCO = area of OACO + area of ABCA

$$= \frac{1}{2}.4.4 + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\, dx$$

$$= 8 + \left[\{\frac{x}{2}\sqrt{32 - x^2} + 16\, sin^{-1}\frac{x}{4\sqrt{2}}\}\right]_4^{4\sqrt{2}}$$

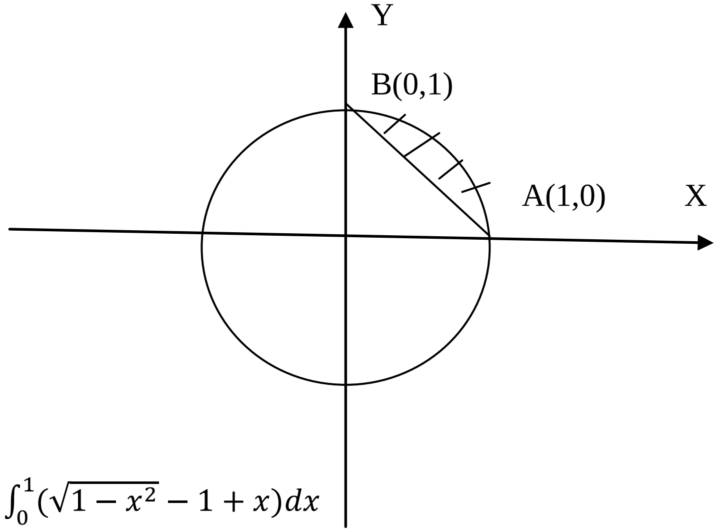
$= 8$ +(0-8) +16($\frac{\pi}{2} - \frac{\pi}{4}$)

$= 4\pi$ Sq. units

LEVEL III

Q1. Find the area of the region : $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$

Sol.



Req area = $\int_0^1 (\sqrt{1-x^2} - 1 + x)dx$

$= \left[\{\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}sin^{-1}x - x + \frac{x^2}{2}\}\right]_0^1$

$= \left(\frac{1}{2} \times \frac{\pi}{2} - 1 + \frac{1}{2}\right) = (\frac{\pi}{4} - \frac{1}{2})\ Sq.units$

(v) Area of the region enclosed between Circle and parabola

LEVEL III

Q2. Find the area lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

Sol.The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as

$$(x - 4)^2 + y^2 = 16.$$

Thus, the centre of the circle is $(4, 0)$ and radius is 4.

Its intersection with the parabola $y^2 = 4x$ gives

$$x^2 + 4x = 8x$$

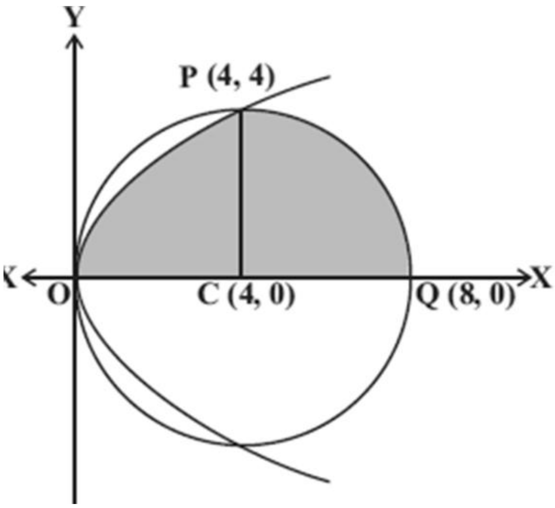
$$\text{or } x^2 - 4x = 0$$

$$\text{or } x(x - 4) = 0$$

$$\text{or } x = 0, \quad x = 4$$

Thus, the points of intersection of these two curves are $O(0, 0)$ and $P(4,4)$ above the x axis. From theFig , the required area of the region OPQCO included between the two curves above x – axis is

= (area of the region OCPO) + (area of the region PCQP)



$$= \int_0^4 y dx + \int_4^8 y\, dx$$

$$= 2\int_0^4 \sqrt{x} dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\, dx$$

$$= 2.\frac{2}{3}\left[\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]_0^4 + \left[\frac{x}{2}\sqrt{4^2 - (x-4)^2} + 8\, sin^{-1}\frac{(x-4)}{4}\right]_4^8$$

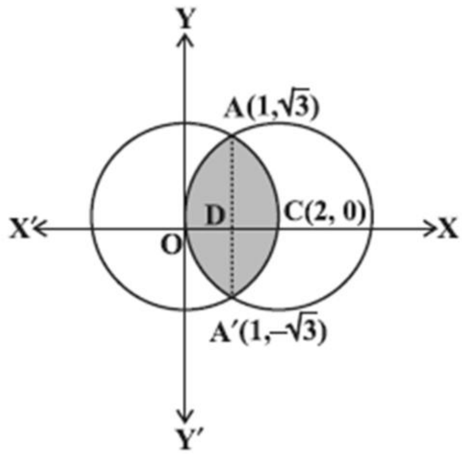
$$= \frac{32}{3} + \left[\frac{4}{2}.0 + \frac{1}{2}.4^2.sin^{-1} 1\right] = \frac{32}{3} + \left[0 + 8.\frac{\pi}{2}\right] = \left(\frac{32}{3} + 4\pi\right) Sq.units$$

(vi) Area of the region enclosed between Two Circles

LEVEL III

Q1. Find the area bounded by the curves $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$ using integration.

Sol.Equations of the given circles are $x^2 + y^2 = 4$ ... (1)
and $(x – 2)^2 + y^2 = 4$ ... (2)
Equation (1) is a circle with centre O at the origin and radius 2. Equation (2) is a circle with centre C (2, 0) and radius 2.
Solving equations (1) and (2), we have
$(x –2)^2 + y^2 = x^2 + y^2$
or$x^2 – 4x + 4 + y^2 = x^2 + y^2$ or x = 1 which gives y = ± 3
Thus, the points of intersection of the given circles are A(1, 3) and A′(1, – 3) as shown in the Fig



Required area of the enclosed region OACA′O between circles
= 2 [area of the region ODCAO]
= 2 [area of the region ODAO + area of the region DCAD]

$$= 2\{\int_0^1 y dx + \int_1^2 y dx\}$$

$$= 2\{\int_0^1 \sqrt{4-(x-2)^2} dx + \int_1^2 \sqrt{4-x^2} dx\}$$

$$= 2\left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2} + \frac{1}{2}.4.sin^{-1}\frac{(x-2)}{2}]\right]_0^1 + 2\left[\frac{x}{2}\sqrt{4-x^2} + \frac{1}{2}.4.sin^{-1}\frac{x}{2}]\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4.sin^{-1}\frac{(x-2)}{2}]\right]_0^1 + \left[x\sqrt{4-x^2} + 4.sin^{-1}\frac{x}{2}]\right]_1^2$$

$$= \left[-\sqrt{3} + 4\,sin^{-1}(-\frac{1}{2}\right) - 4\,sin^{-1}(-1)] + [4\,sin^{-1}1 - \sqrt{3} - 4\,sin^{-1}(\frac{1}{2})]$$

$$= \left[\left(-\sqrt{3} - 4.\frac{\pi}{6}\right) + 4.\frac{\pi}{2}\right] + \left[4.\frac{\pi}{2} - \sqrt{3} - 4.\frac{\pi}{6}\right]$$

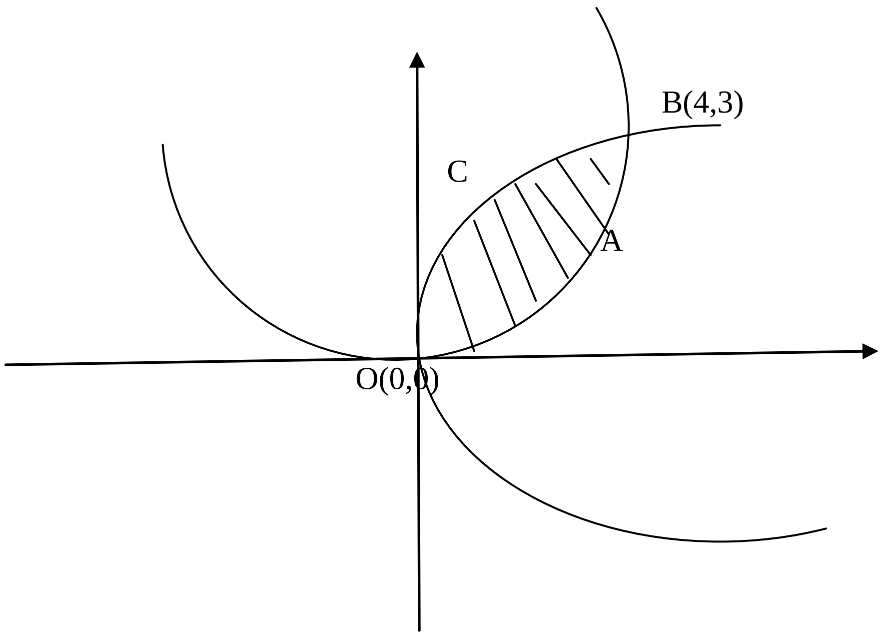
$= \frac{8\pi}{3} - 2\sqrt{3}$ Sq. units

(vii) Area of the region enclosed between Two parabolas

LEVEL II

Q1. Draw the rough sketch and find the area of the region bounded by two parabolas $4y^2 = 9x$ and $3x^2 = 16y$ by using method of integration.

Sol.



The point of intersection of curve $3x^2 = 16y$ &$4y^2 = 9x$ are (0,0) &(4,3)

Required area = area of OABCO

$$= \int_0^4 (y_{Upper} - y_{Lower})\, dx$$

$$= \int_0^4 \left(\frac{3}{2}\sqrt{x} - \frac{3}{16}x^2\right) dx$$

$$= \left[\frac{3}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3}{16}\frac{x^3}{3}\right]_0^4 = \left(4^{\frac{3}{2}} - 0\right) - \frac{1}{16}(4^3 - 0)$$

$$= 8 - 4 = 4\ sq.units$$

(viii) Area of triangle when vertices are given

LEVEL III

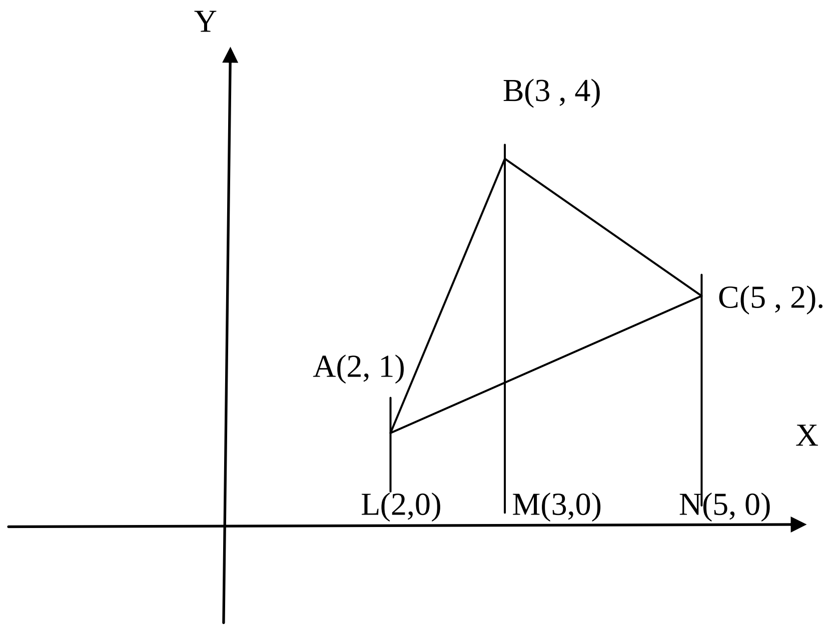
Q1. Using integration compute the area of the region bounded by the triangle whose vertices are (2 , 1), (3 , 4), and (5 , 2).

Sol. Let A(2, 1) , B(3 , 4) and C(5 , 2).

Equation of AB is $\frac{y-1}{x-2} = \frac{4-1}{3-2}$

$$\Rightarrow \frac{y-1}{x-2} = 3 \ \Rightarrow y = 3x - 5$$

Similarly, Equation of BC is $y = 7 - x and Equation of CA is y = \frac{x+1}{3}$



Required area of $\Delta ABC = \int_2^3 y_{AB} dx + \int_3^5 y_{BC}\, dx - \int_2^5 y_{CA}\, dx$

$=\int_2^3 (3x - 5)\, dx + \int_3^5 (7 - x)\, dx - \int_2^5 \frac{x+1}{3} dx$

$$= \left[\left(3\frac{x^2}{2} - 5x\right)\right]_2^3 + \left[\left(7x - \frac{x^2}{2}\right)\right]_3^5 - \frac{1}{3}\left[\left(\frac{x^2}{2} + x\right)\right]_2^5$$

$$= \left[\left(3.\frac{3^2}{2} - 15\right) - \left(3.\frac{2^2}{2} - 10\right)\right] + \left[\left(35 - \frac{25}{2}\right) - \left(21 - \frac{9}{2}\right)\right] - \frac{1}{3}[(\frac{5^2}{2} + 5) - (\frac{2^2}{2} + 2))$$

$$= \left[\left(\frac{27}{2} - 15\right) - (6 - 10)\right] + \left[\left(35 - \frac{25}{2}\right) - \left(21 - \frac{9}{2}\right)\right] - \frac{1}{3}[\left(\frac{25}{2} + 5\right) - (2 + 2)]$$

$$= 4\ sq.units$$

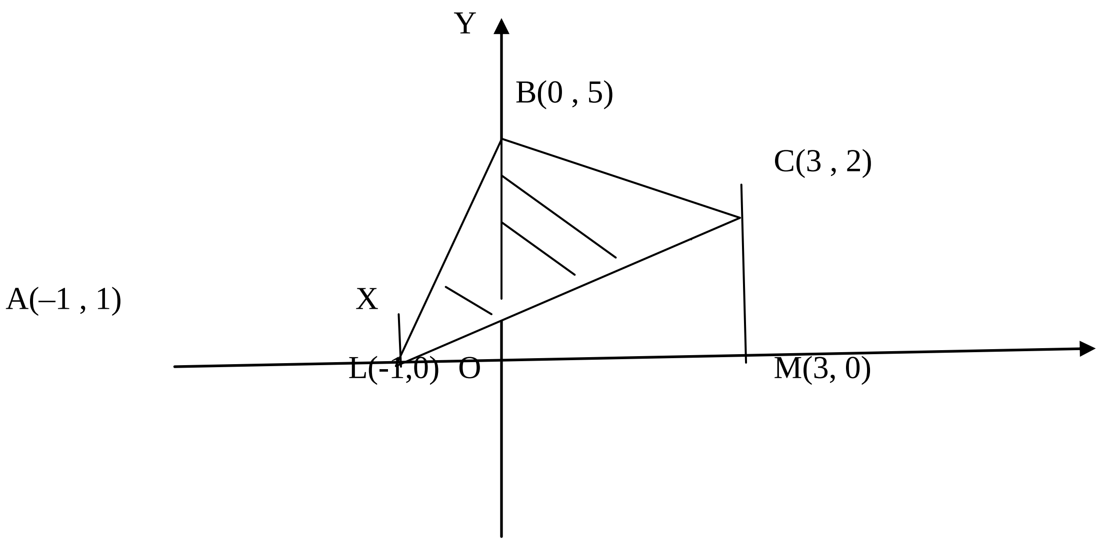
Q2. Using integration compute the area of the region bounded by the triangle whose vertices are (–1 , 1), (0 , 5), and (3 , 2).

Sol. Let A(–1 , 1), B(0 , 5) and C(3 , 2) are three given vertices of a triangle as shown in the figure.

Equation of AB $\frac{y-1}{x+1} = \frac{5-1}{0+1}$

$\therefore y = 4x + 5$

Equation of BC $3y = 15 - 3x$ and Equation of CA $y = \frac{x}{4} + \frac{5}{4}$



Required area of $\Delta ABC = \int_{-1}^{0} y_{AB} dx + \int_{0}^{3} y_{BC}\, dx - \int_{-1}^{3} y_{CA}\, dx$

$$= \int_{-1}^{0} (4x + 5) dx + \int_{0}^{3} (5 - x)\, dx - \int_{-1}^{3} \left(\frac{x}{4} + \frac{5}{4}\right) dx$$

$$= + -$$

$$= \left[\left(4\frac{x^2}{2} + 5x\right)\right]_{-1}^{0} + \left[\left(5x - \frac{x^2}{2}\right)\right]_{0}^{3} - \left[\left(\frac{x^2}{8} - \frac{5}{4}x\right)\right]_{-1}^{3}$$

$$= [(2x^2 + 5x)]_{-1}^{0} + \left[\left(5x - \frac{x^2}{2}\right)\right]_{0}^{3} - \left[\left(\frac{x^2}{8} - \frac{5}{4}x\right)\right]_{-1}^{3}$$

$$= (0 + 0 - 2 + 5) + \left(15 - \frac{9}{2} - 0 - 0\right) - \left(\frac{9}{8} + \frac{15}{4} - \frac{1}{8} + \frac{5}{4}\right)$$

$= (0 + 0 - 2 + 5) + \frac{21}{2} - (\frac{9 + 30 - 1 + 10}{8})$

$= 3 + \frac{21}{2} - \frac{48}{8} = 3 + \frac{21}{2} - 6 = \frac{15}{2} Sq.Units$

(ix) Area of triangle when sides are given

LEVEL III

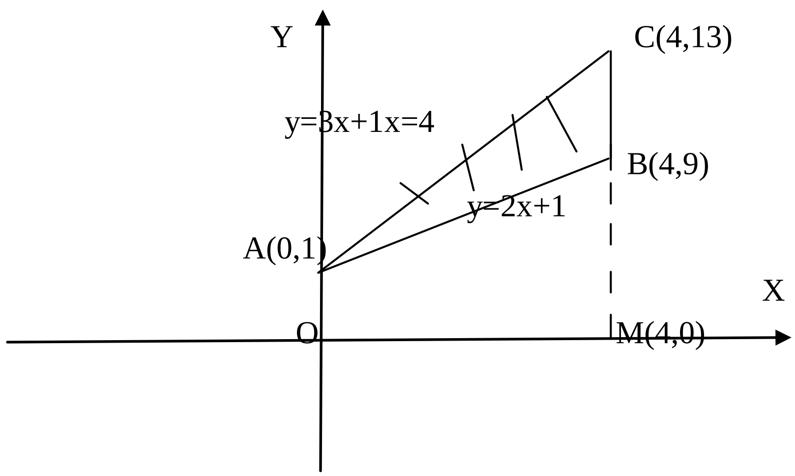
Q1. Using integration find the area of the region bounded by the triangle whose sides are y = 2x + 1, y = 3x + 1, x = 4.

Sol. Here y = 2x + 1…………(1)

y = 3x + 1…………(2)

x = 4……………(3)

Solving (1) & (3), we get (4, 9) , Solving (1) & (2),we get (0, 1) & Solving (2) & (3) we get (4, 13)



Required area of $\Delta ABC = \int_0^4 y_{AC} dx - \int_0^4 y_{AB}\, dx$

$$= \int_0^4 (3x + 1)dx - \int_0^4 (2x + 1)\, dx$$

$=\left[\left(3\frac{x^2}{2} + x\right)\right]_0^4 - \left[\left(2\frac{x^2}{2} + x\right)\right]_0^4$

$= [(3\frac{4^2}{2} + 4)-(4^2 + 4)] = (24 + 4) - (16 + 4) = 8\ sq.units$

Q2. Using integration compute the area of the region bounded by the lines

x + 2y = 2, y – x =1, and 2x + y = 7.

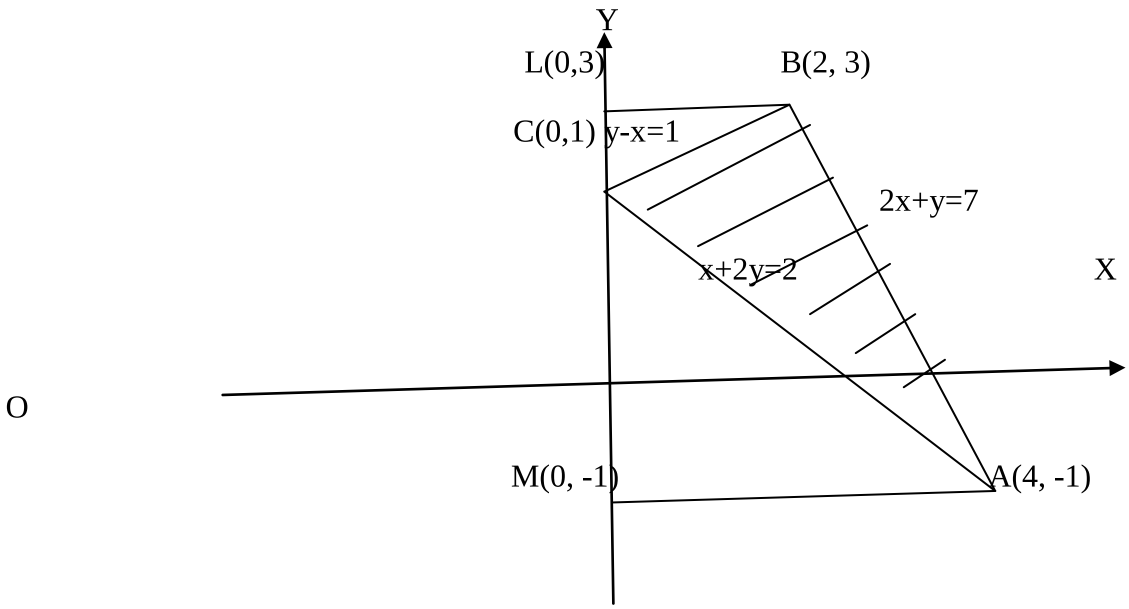
Sol. Here, x + 2y = 2…………(1)

y – x =1……………(2)

2x + y = 7. …………(3)

Solving (1) ,(2) and (3), we get

A(4, -1), B(2, 3) and C(0,1)



Required area of $\Delta ABC = \int_{-1}^{3} X_{BC} dy - \{\int_{1}^{3} X_{BC}\, dy + \int_{-1}^{1} X_{AC} dy\}$

$= \int_{-1}^{3} \left(\frac{7-y}{2}\right) dy - \{\int_{1}^{3} (y-1)\, dy + \int_{-1}^{1} (2-2y) dy\}$

$$= \frac{1}{2}\left[\left(7y - \frac{y^2}{2}\right)\right]_{-1}^{3} - \{\left[\left(\frac{y^2}{2} - y\right)\right]_{1}^{3} + [(2y - y^2)]_{-1}^{1}\}$$

$$= \frac{1}{2}\left[\left(\frac{21}{2} - \frac{9}{2}\right) - \left(-7 - \frac{1}{2}\right)\right] - \{\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) + [(2-1) - (-2-1)]$$

$= \frac{1}{2}\left(\frac{33}{2} + \frac{15}{2}\right) - \left(\frac{3}{2} + \frac{1}{2}\right) + 4 = 12 - (2+4) = 6 Sq.Units$

TOPIC- 8

DIFFERENTIAL EQUATIONS

1. Order and degree of a differential equation

LEVEL I

Q1. Write the order and degree of the following differential equations

(i) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 2y = 0$

Sol. Order of Diff equation is =2

Degree ofDiff equation is =2

2. General and particular solutions of a differential equation

LEVEL I

Q1. Show that $y = e^{-x} + ax + b$ is the solution of $e^x \frac{d^2y}{dx^2} = 1$

Sol. We have,

$y = e^{-x} + ax + b$……..(1)

Differentiate (1) w.r.t x , we get

$$\frac{dy}{dx} = -e^{-x} + a$$

Again, Differentiate,

$$\frac{d^2y}{dx^2} = e^{-x} \implies e^x \frac{d^2y}{dx^2} = 1$$

3. Formation of differential equation

LEVEL II

Q1 Obtain the differential equation by eliminating a and b from the equation $y = e^x(a\cos x + b\sin x)$

Sol.The equation of curve is

$y=e^x(a\cos x+b\sin x)$

Differentiating y w.r.t. x , we get

$y_1=e^x(-a\sin x + b\cos x) + e^x(a\cos x+b\cos x)$

Or y_1 - y $=e^x(-a\sin x+b\cos x)$ --------------- (1)

Differentiating wrt x once again we get

$y_2 - y_1=e^x(-a\cos x-b\sin x) + e^x(-a\sin x+b\cos x)$

$=e^x(a\cos x+b\sin x) + y_1 - y$ (from i)

Or y_2 - y_1 = - y + y_1 – y

Or y_2 - $2y_1$ +2y = 0 or $\frac{d^2y}{dx^2}$ -2$\frac{dy}{dx}$+2y = 0

LEVEL III

Q 1. Find the differential equation of the family of circles $(x - a)^2 - (y - b)^2 = r^2$

Sol. The given curve is $(x-a)^2+ (y-b)^2 = r^2$

Differentiating wrt x, we get

$2(x-a)+ 2(y-b)y_1 = 0$

$y_1=-\frac{(x-a)}{y-b}$

$y_2= - [\frac{((y-b)-y_1(x-a))}{(y-b)^2}]$

$y_2=- [\frac{((y-b)+(\frac{(x-a)(x-a)}{y-b}))}{(y-b)^2}]$ from (1)

$y_2=- [\frac{((y-b)^2+(x-a)^2))}{(y-b)^3}] = \frac{-r^2}{(y-b)^3}$

Now $1+{y_1}^2=1+\frac{(x-a)^2}{(y-b)^2} =\frac{((y-b)^2+(x-a)^2))}{(y-b)^3}$

$1+{y_1}^2 =\frac{r^2}{(y-b)^2}$

Or $(1+{y_1}^2)^3 =[\frac{r^2}{(y-b)^2}]^3=\frac{r^6}{(y-b)^6}$ -------------------- (3)

${y_2}^2=\frac{r^4}{(y-b)^6}$ from (2) --------------------------------- (4)

From (3) and (4) we get

$\frac{(1+{y_1}^2)^3}{{y_2}^2} =\frac{r^6}{r^4}$ or $(1+{y_1}^2)^3=r^2{y_2}^2$

Q2. Obtain the differential equation representing the family of parabola having vertex at the origin and axis along the positive direction of x-axis

Sol.The equation of parabola may be taken as

y^2=4ax ------ (1)

Where a is parameter

Differentiating wrt x we get

2yy_1= 4a -------- (2)

From (1) and (2) we get

$\frac{y^2}{2yy_1}=\frac{4ax}{4a}$

$\frac{y}{2y_1}$=x or y = $2xy_1$

2x$\frac{dy}{dx}$ = y

4. Solution of differential equation by the method of separation of variables

LEVEL II

Q1. Solve $\frac{dy}{dx} = 1 + x + y + xy$

Sol.The differential equation is

y_1=(1+x)(1+y) or $\frac{dy}{1+y}$ = (1+x) dx

Integrating both sides we get

$\int \frac{dy}{1+y} = \int (1+x)\, dx + c$

$\log|1+y| = x + \frac{x^2}{2} + c$

Q2. Solve $\frac{dy}{dx} = e^{-y} \cos x$ given that y(0)=0.

Sol. The given differential equation is

$\frac{dy}{dx} = e^{-y} \cos x$ or $e^{y} dy = \cos x\, dx$

Integrating both sides , we get

$\int e^{y} dy = \int \cos x\, dx + c$

$e^{y} = \sin x + c$

Given that x=0 , y=0 So $e^{0} = \sin 0 + c$,c=1

$e^{y} = \sin x + 1$

Q3. Solve $(1+x^2)\frac{dy}{dx} - x = \tan^{-1} x$

Sol. The given differential equation is

$(1+x^2)\frac{dy}{dx} - x = \tan^{-1} x$

Or $\frac{dy}{dx} = \frac{x+\tan^{-1} x}{1+x^2}$ or $dy = \frac{x+\tan^{-1} x}{1+x^2}\, dx$

Integrating both sides , we get

$\int dy = \int \frac{x+\tan^{-1} x}{1+x^2}\, d\,x + c$

$Y = \int \frac{x}{1+x^2}\, d\,x + \int \frac{\tan^{-1} x}{1+x^2}\, d\,x + c$

$= \frac{1}{2} \log(1+x^2) + \frac{1}{2} (\tan^{-1} x)^2 + c$

$2y = \log(1+x^2) + (\tan^{-1} x)^2 + c$

5 .Homogeneous differential equation of first order and first degree

LEVEL II

Q1. Solve $(x^2+xy)dy=(x^2+y^2)dx$

Sol. $The given d.e. is \ (x^2+xy)dy = (x^2+y^2)dx$

$$or \frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$$

$$Let y = xv, then \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$So x\frac{dv}{dx} + v = \frac{x^2+x^2v^2}{x^2+x^2v} = \frac{1+v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1-v}{1+v}$$

$$(\frac{1+v}{1-v})dv = \frac{dx}{x} or \frac{-(1-v)+2}{1-v}dv = \frac{dx}{x}$$

$$Or\ (-1 + \frac{2}{1-v})dv = \frac{dx}{x}$$

$Integrating both sides$

$$\int (-1 + \frac{2}{1-v})dv = log|x| + c$$

$$-v-2\,log|1-v| = log|x| + c$$

$$Or \quad -\frac{y}{x} - 2\,log\left|1-\frac{y}{x}\right| = log|x| + c$$

$$-\frac{y}{x} - 2\,log|x-y| + 2log|x| = log|x| + c$$

$$\frac{y}{x} - 2\,log|x-y| + log|x| = c$$

LEVEL III

Show that the given differential equation is homogenous and solve it.

$$Q1.\ (x-y)\frac{dy}{dx} = x+2y$$

Sol. The given differential equation is $(x-y)\frac{dy}{dx} = x+2y$

$$put\ y = vx \implies \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$So\ x\frac{dv}{dx} + v = \frac{1+2v}{1-v}$$

$$\implies \frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

Integrating both sides we get

$$\int \frac{1-v}{1+v+v^2}dv = \int \frac{dx}{x} + c$$
$$\int \frac{-1/2(2v+1)}{1+v+v^2}dv + \frac{3}{2}\int \frac{dv}{1+v+v^2} = \int \frac{dx}{x} + c$$
$$\Rightarrow -\frac{1}{2}log(1+v+v^2) + \frac{3}{2}\int \frac{dv}{(v+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = logx + c$$
$$\Rightarrow -\frac{1}{2}log\left(1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\right) + \frac{3}{2}.\frac{1}{\sqrt{3}/2}tan^{-1}\left(\frac{v+1/2}{\sqrt{3}/2}\right) = logx + c$$
$$\Rightarrow -\frac{1}{2}log(x^2+xy+y^2) + \frac{1}{2}logx^2 + \sqrt{3}\,tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = logx + c$$
$$\Rightarrow -\frac{1}{2}log(x^2+xy+y^2) + logx + \sqrt{3}\,tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right) = logx + c$$
$$\Rightarrow -\frac{1}{2}log(x^2+xy+y^2) + \sqrt{3}\,tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = c$$

Q2. $ydx + x\log(\frac{y}{x})dy - 2xdy = 0$

Sol. $wehave, ydx + xlog\left(\frac{y}{x}\right)dy - 2xdy = 0$

$$\frac{dy}{dx} = \frac{y}{2x - xlog\left(\frac{y}{x}\right)}$$

$puty = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\Rightarrow x\frac{dv}{dx} + v = \frac{v}{2-logv}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2-logv} - v \Rightarrow x\frac{dv}{dx} = \frac{-v+vlogv}{2-logv}$$
$$\Rightarrow \frac{2-logv}{-v+vlogv}dv = \frac{dx}{x}$$

Integrating both sides we get

$$\int \frac{2-logv}{-v+vlogv}dv = \int \frac{dx}{x}$$
$$\Rightarrow \int \frac{2dv}{v(logv-1)} - \int \frac{logvdv}{vlogv-v} = \int \frac{dx}{x} + \text{logc}$$
$$2\,log(logv-1) - log\,(vlogv-v) = \text{logcx}$$
$$\Rightarrow log\,(logv-1)^2 - log(v(logv-1)) = logcx$$

$$\Rightarrow log\left(\frac{(logv-1)^2}{v(logv-1)}\right) = logcx$$

$$\Rightarrow \frac{logv-1}{v} = cx \Rightarrow logv = 1 + cvx$$

$$\Rightarrow log\frac{y}{x} = 1 + cy$$

$Q3. Solve$ $xdy - ydx = \sqrt{x^2 - y^2}dx$

$Sol.$ The given differential equation is

$xdy - ydx = \sqrt{(x^2 - y^2)}dx$

$xdy = (y + \sqrt{(x^2 - y^2)})dx$

$$\frac{dy}{dx} = \frac{y + \sqrt{(x^2 - y^2)}}{x}$$

$$Let y = xv \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$So x\frac{dv}{dx} + v = v + \sqrt{1 - v^2}$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x}$$

Integrating both sides we get

$$\int \frac{dv}{\sqrt{1 - v^2}}dv = \int \frac{dx}{x} + C$$

$sin^{-1} v = log|x| + c$

$$sin^{-1}\frac{y}{x} = log\, x + c$$

Q4.Solve $x^2 ydx - (x^3 + y^3)dy = 0$

Sol.The given differential equation is

$x^2 ydx = (x^3 + y^3)dy$

$$Or \frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

$$Let y = xv \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

So we get $v + x\frac{dv}{dx} = \frac{x^2 xv}{x^3 + x^3 v^3}$

$$v + x\frac{dv}{dx} = \frac{v}{1 + v^3}$$

$$x\frac{dv}{dx} = \frac{v}{1 + v^3} - v = \frac{v - v - v^4}{1 + v^3} == \frac{-v^4}{1 + v^3}$$

$$or \frac{1 + v^3}{v^4}dv = \frac{-dx}{x}$$

$(\frac{1}{v^4}+\frac{1}{v})dv = \frac{-dx}{x}$

Integrating both sides we get

$\int (\frac{1}{v^4}+\frac{1}{v})dv = -\int \frac{dx}{x} + logC$

$\frac{v^{-3}}{-3} + \; log\, v = \; - \; log\, x \; + \; logc$

$\frac{1}{-3}\frac{x^3}{y^3} \; + \; log\frac{y}{x} \; = \; - \; log\, x \; + \; c$

$\frac{1}{-3}\frac{x^3}{y^3} \; + \; log|y| \; - \; log\, x \; = - \; log\, x \; + \; logc$

$log \; \frac{y}{c} \; = \; \frac{1}{3} \cdot \frac{x^3}{y^3}$

$y \; = ce^{\frac{1x^3}{3y^3}}$

Q5. Solve $xdy - ydx = \sqrt{(x^2+y^2)}dx$

Sol. $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

Or, $\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left((\frac{y}{x})^2\right)}$

$Let\; y = xv \Rightarrow \frac{dy}{dx} \; = x\frac{dv}{dx} + v$

$\Rightarrow x\frac{dv}{dx} + v = v + \sqrt{1+v^2}$

$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating both sides we get

$\Rightarrow log\left(v+\sqrt{1+v^2}\right) = logx + logc$

$\Rightarrow log\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right) = logcx$

$\Rightarrow y + \sqrt{x^2+y^2} = cx^2$

Q6.Solve $\left(\mathrm{y} \; + \; 3\mathrm{x}^2\right)\frac{dx}{dy} \; = \; \mathrm{x}$

Sol. $\frac{dy}{dx} = \frac{y}{x} + 3x$

$Let\; y = xv \Rightarrow \frac{dy}{dx} \; = x\frac{dv}{dx} + v$

$\Rightarrow v + x\frac{dv}{dx} = v + 3x$

$\Rightarrow x\dfrac{dv}{dx} = 3x$

$\Rightarrow dv = 3dx$

Integrating both sides we get

$\Rightarrow v = 3x + c$

$\Rightarrow \dfrac{y}{x} = 3x + c$

$\Rightarrow y = 3x^2 + cx$

Q7. Solve $x\ dy + (y - x^3)dx = 0$

Sol. $\frac{dy}{dx} = \frac{x^3-y}{x} = x^2 - \frac{y}{x}$

$\Rightarrow \dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

Here P $= \frac{1}{x}$ and Q $= x^2$ So IF $= e^{\int pdx} = e^{\int\frac{1}{x}dx} = e^{\log x} = x$

Solution is given by

$Y(I.F.) = \int Q(I.F.)\,dx + C$

$yx = \int x^2.xdx + C$

$\Rightarrow yx = \dfrac{x^4}{4} + C$

$\Rightarrow y = \dfrac{x^3}{4} + \dfrac{C}{x}$

Q8. Solve $xdy + (y + 2x^2)dx = 0$

Sol. $\frac{dy}{dx} + \frac{y}{x} = -2x$

$Here\ P = \frac{1}{x} and Q = -2x So IF = e^{\int pdx} = e^{\int\frac{1}{x}dx} = e^{logx} = x$

Solution is given by

$Y(I.F.) = \int Q(I.F.)\,dx + C$

$yx = \int (-2x).xdx + C$

$yx = -\dfrac{2}{3}x^3 + C$

$\Rightarrow y = -\dfrac{2}{3}x^2 + \dfrac{C}{x}$

6. Linear Differential Equations

LEVEL I

$Q1.$ Find the integrating factor of the differential $x\dfrac{dy}{dx} - y = 2x^2$

Sol.The given equation is $xy_1 - y = 2x^2$

$$ORy_1 + (-\frac{1}{x})y = 2x,$$

$$\text{Here P} = -\frac{1}{x} \text{ and Q} = 2x \text{ So IF} = e^{\int pdx} = e^{\int -\frac{1}{x}}$$

$$IF = e^{log x^{-1}} = x^{-1} = \frac{1}{x}$$

LEVEL II

Q1. $Solve \frac{dy}{dx} + 2y\tan x = \sin x$

The given differential equation is

So

$$\frac{dy}{dx} + (2\tan x)y = \sin x + c$$

$HereP = 2 \tan x , Q = \sin x$

$SoI.F. = e^{\int Pdx} = e^{\int 2\tan x dx} = e^{2\log|\sec x|}$

$$= e^{\log(\sec x)^2} = (\sec x)^2$$

Solution is given by

$Y(I.F.) = \int Q(I.F.)\,dx + C$

$y(\sec x)^2 = \int \sin x \sec^2 x dx + C$

$y(\sec x)^2 = \int \tan x \sec x\, dx + C$

$y(\sec x)^2 = \sec x + C$

Q2.Solve $(1+x)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Sol. The given differential equation is

$$(1+x)\frac{dy}{dx} - y = e^{3x}(1+x)^2$$

$$Or \frac{dy}{dx} + (\frac{-1}{1+x})y = e^{3x}(1+x)$$

$$SoI.F. = e^{\int Pdx} = e^{\int(-\frac{1}{1+x})dx} = e^{-\log(1+x)} = (1+x)^{-1} = \frac{1}{1+x}$$

Solution is given by

$Y(I.F.) = \int Q(IF)\,dx + cORy(\frac{1}{1+x}) = \int e^{3x}(1+x) \quad \frac{1}{1+x}dx + C$

$= \int e^{3x}\,dx + C$

$$\frac{y}{1+x} = \frac{1}{3}e^{3x} + C$$

Q3. Solve $x\frac{dy}{dx}+y=x\log x$

Sol.The given differential equation is

$$x\frac{dy}{dx} + y = x\log x \; OR \; \frac{dy}{dx} + \frac{y}{x} = \log x$$

$$Here p = \frac{1}{x}, Q = \log x \; SoI.F. = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Solution is given by

$$Y(I.F.) = \int Q(IF)\, dx + c$$

$$Yx = \int x \log x \, dx + c$$

$$Yx = \log x \int x\, dx - \int (\frac{1}{x}\int x\, dx) dx + C$$

$$xy = \frac{x^2}{2}\log x - \int \frac{1}{x}\frac{x^2}{2} dx + C = \frac{x^2}{2}\log x - \frac{x^2}{4} + C$$

LEVEL III

Q1Solve $\frac{dy}{dx}=\cos(x+y)$

Sol The given differential equation is

$$\frac{dy}{dx} = \cos(x+y)$$

$$Let v = x + y or \frac{dv}{dx} = 1 + \frac{dy}{dx} or \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$Or \frac{dv}{dx} - 1 = (\cos v)$$

$$\frac{dv}{1+\cos(v)} = dx or \frac{1}{2}\sec^2\frac{v}{2} dv = dx$$

Integrating both sides , we get

$$\frac{1}{2}\int \sec^2\frac{v}{2} dv = x + c$$

$$\frac{1}{2}.2\tan\frac{v}{2} = x + c$$

$$Or \tan\frac{x+y}{2} = x + c$$

Q2.Solve $ye^{y}dx=(y^3+2xe^{y})dy$

Sol The given differential equation is

.

$$ye^y dx = (y^3 + 2xe^y)dy$$

$$\frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y} = \frac{2x}{y} + \frac{y^2}{e^y}$$

$$\frac{dx}{dy} + (-\frac{2}{y})\, x = y^2 e^{-y}$$

$$Here P = -\frac{2}{y}, Q = y^2 e^{-y}$$

$$SoI.F. = e^{\int P dy} = e^{\int -\frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}}$$

$$I.F. = y^{-2} = \frac{1}{y^2}$$

Solution is given by

$$(IF)x = \int Q(IF)dy + C$$

$$\frac{1}{y^2} x = \int y^2 e^{-y} \frac{1}{y^2} dy + C$$

$$\frac{x}{y^2} = \int e^{-y} dy + C = - e^{-y} + C$$

$$x = -y^2 e^{-y} + cy^2$$

Q3.Solve $x^2 \frac{dy}{dx} = y(x + y)$

$Sol.$ The given differential equation is

$$\frac{x^2 dy}{dx} = y(x + y)$$

$$Or \frac{dy}{dx} = \frac{xy + y^2}{x^2}$$

$$Let y = xv \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$Or v + x\frac{dv}{dx} = \frac{xxv + x^2 v^2}{x^2} = v + y^2$$

$$x\frac{dv}{dx} = v^2 or \frac{dv}{v^2} = \frac{dv}{x}$$

Integrating both sides , we get

$$\int v^{-2} dv = log|x| + c or \quad -\frac{1}{v} = log|x| + c$$

$$-\frac{x}{y} = log|x| + c$$

Q4.Solve $\frac{dy}{dx} + \frac{4x}{x^2 + 1} y = -\frac{1}{(x^2 + 1)^3}$

Sol.

The given differential equation is $\frac{dy}{dx} + (\frac{4x}{x^2 + 1})y = -\frac{1}{(x^2 + 1)^3}$

$Here\ p = \frac{4x}{x^2+1}\ , Q\ =\ -\frac{1}{(x^2+1)^3}$

$SoI.F. =\ e^{\int Pdy}\ = e^{\int \frac{4x}{x^2+1}dy}\ = e^{2log(x^2+1)} = e^{log((x^2+1)^2)}$

$I.F. =\ (x^2+1)^2$

Solution is given by

$y(IF) = \int Q(IF)dx + C$

$$y(x^2+1)^2 = \int -\frac{(x^2+1)^2}{(x^2+1)^3}dx\ + c$$

$$y(x^2+1)^2 = \int -\frac{1}{(x^2+1)}dx\ + c$$

$$y(x^2+1)^2\ = -\ tan^{-1}x\ + c$$

Q5 Solve the differential equation $(x+2y^2)\frac{dy}{dx} = y$;given that when x=2,y=1

Sol.*The equation is*

$(x+2y^2)\ \frac{dy}{dx}\ =\ y$

$x + 2y^2 = y.\frac{dx}{dy}$

$\frac{x}{y} + 2y = \frac{dx}{dy}$

$\frac{dx}{dy} - \frac{x}{y}\ =\ 2y$

$HereP\ =\ -\frac{1}{y}\ , Q\ =\ 2y$

$SoIF\ =\ e^{\int -\frac{1}{y}dy}\ =\ e^{-log|y|} = e^{log\, y^{-1}}\ = y^{-1}\ = \frac{1}{y}$

Solution is given by

$x(IF) = \int Q(IF)dx + C$

$\frac{1}{y}x\ =\ \int \frac{1}{y}.2ydy + C\ =\ 2y\ + c$

$Given\ that\ \ x = 2, y = 1, so\ 2 = 2 + c\ , C = 0$

$\frac{x}{y} = 2y\ \ Or\ \ x = 2y^2$

TOPIC-9

VECTOR ALGEBRA

(i) Vector and scalars, Direction ratio and direction cosines & Unit vector

LEVEL I

Q1 Sol: $\vec{a} = \hat{\imath} + \hat{\jmath} - 5\hat{k}$;$\vec{b} = \hat{\imath} - 4\hat{\jmath} + 3\hat{k}$

Now, $\vec{a} + \vec{b} = (\hat{\imath} + \hat{\jmath} - 5\hat{k})$+$(\hat{\imath} - 4\hat{\jmath} + 3\hat{k}) = 2\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$

$$\left|\vec{a} + \vec{b}\right| = \sqrt{2^2 + (-3)^2 + (-2)^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$$

Unit vector ∥ $\vec{a} + \vec{b} = \dfrac{\vec{a} + \vec{b}}{\left|\vec{a} + \vec{b}\right|} = \dfrac{2\hat{\imath} - 3\hat{\jmath} - 2\hat{k}}{\sqrt{17}} = \dfrac{2\hat{\imath}}{\sqrt{17}} - \dfrac{3\hat{\jmath}}{\sqrt{17}} - \dfrac{2\hat{k}}{\sqrt{17}}$

Q2 Sol: $Let\,\vec{a} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

unit vector in the direction of $\vec{a}$

$$= \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Vector of magnitude 15 unit in the direction of $\vec{a} = 15\,\hat{a}$

Now $|\vec{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{1 + 4 + 4} = 3$

∴ Required vector $= 15\,\hat{a} = 15 * \dfrac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3} = 5\left(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right) = 5\hat{\imath} - 10\hat{\jmath} + 10\hat{k}$

Q3Sol:$\vec{a} = i + j - k; \vec{b} = i - j + k\,; \vec{c} = -i + j + k$

$$\vec{a} + \vec{b} + \vec{c} = (i + j - k) + (i - j + k) + (-i + j + k)$$

$$= i + j + k$$

unit vector in the direction of $\vec{a} + \vec{b} + \vec{c} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{\left|\vec{a} + \vec{b} + \vec{c}\right|}$

$$= \frac{i+j+k}{\sqrt{3}} = \frac{\hat{\imath}}{\sqrt{3}} + \frac{\hat{\jmath}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$$

Q4 Sol: $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

unit vector in the direction of $\vec{a}$

$$= \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Q5Sol: Let $\vec{a} = \hat{i} - 2\hat{j}$

unit vector in the direction of $\vec{a}$

$$= \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Vector of magnitude 7 unit in the direction of $\vec{a} = 7\,\hat{a}$

Now $|\vec{a}| = \sqrt{1^2 + (-2)^2 + 0^2} = \sqrt{1+4} = \sqrt{5}$

$$\therefore \text{Required vector} = 7\,\hat{a} = 7 * \frac{\hat{i} - 2\hat{j}}{\sqrt{5}} = \frac{7}{\sqrt{5}}\left(\hat{i} - 2\hat{j}\right)$$

LEVEL II

Q1 Sol: $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$; $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{a} + \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ & $\vec{a} - \vec{b} = -\hat{\jmath} - 2\hat{k}$

$$(\vec{a} + \vec{b})X(\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = \hat{\imath}(-6+4) - \hat{\jmath}(-4-0) + \hat{k}(-2) = \vec{0}$$

$$\Longrightarrow \vec{n} = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$$

Unit vector $\perp$ to both $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{-2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}}{\sqrt{4+16+4}} = \frac{-2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}}{\sqrt{24}} = \frac{-2(\hat{\imath} - 2\hat{\jmath} + \hat{k})}{2\sqrt{6}} = \frac{-(\hat{\imath} - 2\hat{\jmath} + \hat{k})}{\sqrt{6}}$$

Unit Vector ⊥ to both $\vec{a}+\vec{b}$ & $\vec{a}-\vec{b}$ & $of\ \ magnitude\ 6 = \frac{-5(\hat{\imath}-2\hat{\jmath}+\hat{k})}{\sqrt{6}}$

Q2 Sol:Given$|\vec{a}+\vec{b}| = 1$; $|\vec{a}| = 1, |\vec{b}| = 1$

$$\Longrightarrow |\vec{a}+\vec{b}|^2 = 1^2 \ \Longrightarrow \vec{a}^2+\vec{b}^2+2\vec{a}.\vec{b} = 1$$

$$\Longrightarrow \vec{a}^2+\vec{b}^2+2|\vec{a}||\vec{b}|cos\theta = 1$$

$$\Longrightarrow 1+1+2cos\theta = 1$$

$$\Longrightarrow 2cos\theta = -1 \Longrightarrow cos\theta = \frac{-1}{2}$$

Now ,$|\vec{a}-\vec{b}|^2 = \vec{a}^2+\vec{b}^2-2\vec{a}.\vec{b}$

$$= \vec{a}^2+\vec{b}^2-2|\vec{a}||\vec{b}|cos\theta = \vec{a}^2+\vec{b}^2-2*1*1*\frac{1}{2}$$

=1+1+1

$\Longrightarrow\ |\vec{a}-\vec{b}|^2 = 3 \Longrightarrow |\vec{a}-\vec{b}| = \sqrt{3}$

Q3 Sol: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$2\vec{a}-\vec{b}+3\vec{c} = 2(\hat{i} + \hat{j} + \hat{k}) -(4\hat{i} - 2\hat{j} + 3\hat{k}) +3(\hat{i} - 2\hat{j} + \hat{k})$

$= \hat{\imath}-2\hat{\jmath}+2\hat{k}$

Unit vector ∥ $2\vec{a}-\vec{b}+3\vec{c} = \dfrac{2\vec{a}-\vec{b}+3\vec{c}}{|2\vec{a}-\vec{b}+3\vec{c}|}$

$= \frac{\hat{\imath}-2\hat{\jmath}+2\hat{k}}{3}$

=Vector of magnitude 6 unit in the direction of $2\vec{a}-\vec{b}+3\vec{c} = 6\ (2\vec{a}-\widehat{\vec{b}+3}\vec{c})$

$= 6*\frac{\hat{\imath}-2\hat{\jmath}+2\hat{k}}{3}$ =$2(\hat{\imath}-2\hat{\jmath}+2\hat{k}) = 2\hat{\imath}-\hat{\jmath}+4\hat{k}$

LEVEL – III

Q1 Sol: Let l, m, n are the dc′s of line

$$\therefore l = cos\alpha, \qquad m = cos\beta, \;\; n = cos\gamma$$

We know that $l^2 + m^2 + n^2 = 1$

$$\Rightarrow cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

$$\Rightarrow 1 - sin^2\alpha + 1 - sin^2\beta + 1 - sin^2\gamma = 1$$

$$\Rightarrow 3 - sin^2\alpha - sin^2\beta - sin^2\gamma = 1$$

$$\Rightarrow sin^2\alpha + sin^2\beta + sin^2\gamma = 2$$

Q2 Sol: $valueofp = ?$

Given $(\hat{\imath} + \hat{\jmath} + \hat{k})p$ is a unit vector

$Let\ \vec{a} = p\hat{\imath} + p\hat{\jmath} + p\hat{k}$

$\therefore |\vec{a}| = \sqrt{p^2 + p^2 + p^2} = 1 \Rightarrow \sqrt{3}p = 1 \Rightarrow p = \pm\frac{1}{\sqrt{3}}$

Q3 Sol: Let the angle between line &y − axis be θ

Then, $(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}).\hat{\jmath} = \frac{\left|\sqrt{(\sqrt{2})^2 + 1^2 + 1^2}\right|}{|i|cos\theta}$

$$= 1 = (\sqrt{2})^2 . 1\ cos\theta \Rightarrow cos\theta = \frac{1}{2}$$

Q4 Sol: $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$; $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$

$$\vec{a} \parallel \vec{b}$$

∴ dR′s are proportional

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow \frac{3}{1} = \frac{2}{p} \Rightarrow p = \frac{2}{3}$$

(ii)Position vector of a point and collinear vectors

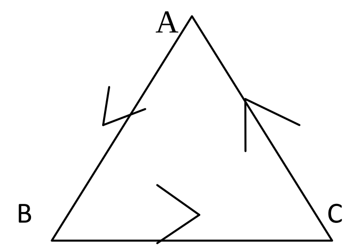
LEVEL – I

Q1 Sol: $\overrightarrow{OA} = 5\hat{\imath} + 3\hat{\jmath}$; $\overrightarrow{OB} = 3\hat{\imath} - \hat{\jmath}$

Let C is the mid pt of AB

Then, $\overrightarrow{OC} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \frac{5\hat{\imath} + 3\hat{\jmath} + 3\hat{\imath} - \hat{\jmath}}{2} = 4\hat{\imath} + \hat{\jmath}$

Q2 Sol: $In\ \Delta\ ABC$



$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$2\hat{\imath} - \hat{\jmath} + 2\hat{k} + \hat{\imath} + 3\hat{\jmath} + 5\hat{k} + \overrightarrow{CA} = 0$

$\Rightarrow \overrightarrow{CA} + 3\hat{\imath} + 2\hat{\jmath} + 7\hat{k} = 0$

$\Rightarrow \overrightarrow{CA} = -(3\hat{\imath} + 2\hat{\jmath} + 7\hat{k})$

Q3 Sol: $Let\ A(1,0),\ B(6,0) \& C(0,0)$

$$\overrightarrow{AB} = 6\hat{\imath} - \hat{\imath} = 5\hat{\imath}\ \ , \overrightarrow{BC} = -6\hat{\imath}\ \ ,\ \ \overrightarrow{CA} = \hat{\imath}$$

$$|\overrightarrow{AB}| = 5, |\overrightarrow{BC}| = 6, |\overrightarrow{CA}| = 1$$

$$\therefore\ |\overrightarrow{AB}| + |\overrightarrow{CA}| = |\overrightarrow{BC}|$$

$\Rightarrow A, B\ and\ C\ are\ \ collinear$

LEVEL – II

Q1 Sol: Given $\overrightarrow{OP} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$; $\overrightarrow{OQ} = -\hat{\imath} + \hat{\jmath} + \hat{k}$

Let R divide PQ in the ratio 2:1 externally

Then $\overrightarrow{OR} = \frac{2(-\hat{\imath}+\hat{\jmath}+\hat{k}) - (\hat{\imath}+2\hat{\jmath}-\hat{k})}{2-1} = = -3\hat{\imath} + 3\hat{k}$

Q2Sol: Given $\overrightarrow{OP} = 2\vec{a} + \vec{b}$; $\overrightarrow{OQ} = \vec{a} - 3\vec{b}$

Let R divide PQ in the ratio 1:2 externally

Then $\overrightarrow{OR} = \frac{\overrightarrow{OQ} - 2\overrightarrow{OP}}{1-2} = \frac{(\vec{a}-3\vec{b}) - 2(2\vec{a}+\vec{b})}{-1} = 3\vec{a} + 5\vec{b}$

Now, $\frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2} = \frac{(\vec{a}-3\vec{b})+(3\vec{a}+5\vec{b})}{2} = \frac{4\vec{a}+2\vec{b}}{2} = 2\vec{a} + \vec{b} = \overrightarrow{OP}$

$\therefore$ P is the mid point of R and Q

(iii) Dot product of two vectors

LEVEL – I

Q1 Sol: $\vec{a}.\vec{b} = (3\hat{\imath} - \hat{\jmath} + 2\hat{k}).(2\hat{\imath} + 3\hat{\jmath} + 3\hat{k}) = 6 - 3 + 6 = 9$

Q2 Sol: $|\vec{a}| = \sqrt{3}$; $|\vec{b}| = 2 \quad \Rightarrow \vec{a}.\vec{b} = \sqrt{6}$

Let θ be the angle between $\vec{a}.\vec{b}$

$$\therefore cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}.2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore cos\theta = \frac{1}{\sqrt{2}} = cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Q3 Sol: $\therefore cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}.2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$$\therefore cos\theta = \frac{1}{\sqrt{2}} = cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

LEVEL – II

Q1 Sol: Let $\vec{a} = x\hat{\imath} + y\hat{\jmath} + 3\hat{k}$

$$\vec{a}\left(\hat{\imath} - 3\hat{k}\right) = 0 \Rightarrow x - 3z = 0 \ldots.(i)$$

$$\vec{a}\left(\hat{\imath} - 2\hat{k}\right) = 5 \Rightarrow x - 2z = 5 \ldots.(ii)$$

$$\vec{a}\left(\hat{\imath} + \hat{\jmath} + 4\hat{k}\right) = 8 \Rightarrow x + y + 4z = 8 \ldots\ldots\ldots\ldots\ldots(iii)$$

Solving (i)&(ii) we get, $\quad x = 15 , z = 5$

Putting in eq (iii) $y = -27$

Then $\vec{a} = 15\hat{\imath} - 27\hat{\jmath} + 5\hat{k}$

Q2 Sol: $Given: \ \left|\vec{a}X\vec{b}\right| = \left|\vec{a}.\vec{b}\right|$

$$\Rightarrow absin\theta = abcos\theta \ \Rightarrow sin\theta = cos\theta$$

$$\Rightarrow tan\,\theta = 1 \Rightarrow \theta = tan^{-1}\,1 = \frac{\pi}{4}$$

$$\therefore \ \theta = \frac{\pi}{4}$$

Q3 Sol: $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$$\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

$$\vec{c} = 3\hat{\imath} + \hat{\jmath}$$

$\vec{a}X\lambda\vec{b} = \left(2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right).\lambda\left(-\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = (2-\lambda)\hat{\imath} + (2+2\lambda)\hat{\jmath} + (3+\lambda)\hat{k}$

$$\left(\vec{a}X\lambda\vec{b}\right) \perp \vec{c}$$

$$\left(\vec{a}X\lambda\vec{b}\right)X\vec{c} = 0$$

$\Rightarrow \left[(2-\lambda)\hat{\imath} + (2+2\lambda)\hat{\jmath} + (3+\lambda)\hat{k}\right].[3\hat{\imath} + \hat{\jmath}] = 0$

$\Rightarrow (2-\lambda)3 + \ (2+2\lambda)1 + (3+\lambda).0 = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow 8 - \lambda = 0 \Rightarrow \lambda = 8$

LEVEL – III

Q1 Sol: $\left|\vec{a} - \vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right|cos\theta$

=1+1-2$cos\theta$=2-2$cos\theta$ =2(1-$cos\theta$)

=2.2$sin^2\frac{\theta}{2} = 4\,sin^2\frac{\theta}{2}$

$$\frac{1}{4}\left|\vec{a} - \vec{b}\right|^2 = sin^2\frac{\theta}{2} \Rightarrow sin\frac{\theta}{2} = \frac{1}{2}\left|\vec{a} - \vec{b}\right|$$

Q2 Sol: $|a + b| = |a - b|$

$$|a + b|^2 = |a - b|^2$$

$a^2 + b^2 + 2abcos\theta = a^2 + b^2 - 2abcos\theta$

$= 4abcos\theta = 0 \Longrightarrow cos\theta = 0 \Longrightarrow \theta = \frac{\pi}{2}$

Q3 Sol: (1) Vector is $\vec{a}\&\vec{b}$ are$\perp$ $if\,\vec{a}.\vec{b} = 0$

$$= \left(3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}\right).\left(\lambda\hat{\imath} - 4\hat{\jmath} + 8\hat{k}\right) = 0$$

$$= 3\lambda + 8 + 32 = 0 \quad \Rightarrow 3\lambda = \ -40 \Rightarrow \lambda = \ \frac{-40}{3}$$

(2) When $\vec{a} \& \vec{b} are$ ∥

$$\vec{a} = \lambda\vec{b} \Rightarrow \lambda = \frac{\vec{a}}{\vec{b}} \Rightarrow \frac{3}{\lambda} = \frac{-2}{-4} = \frac{1}{2}$$

$$\Rightarrow \frac{3}{\lambda} = \frac{1}{2} \Rightarrow \lambda = 6$$

Q4 Sol: Given $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$

$$\Rightarrow \vec{x}^2 - \vec{a}^2 = 15 \Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$$

$$\Rightarrow |\vec{x}|^2 - 1 = 15 \Rightarrow |\vec{x}|^2 = 16 \Rightarrow |\vec{x}| = 4$$

Q5 Sol: $\vec{a} = 5\hat{i} - \hat{j} + 7k$ and $\vec{b} = \hat{i} - \hat{j} + \mu\hat{k}$

$\vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \mu)\hat{k}$

$\vec{a} - \vec{b} = 4\hat{i} + (7 - \mu)\hat{k}$

Given that $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$ are orthogonal

$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

$(6\hat{i} - 2\hat{j} + (7 + \mu)\hat{k}).(4\hat{i} + (7 - \mu)\hat{k}) = 0$

$24 + 49 - \mu^2 = 0$

$\Rightarrow \mu^2 = 73, \Rightarrow \mu = \pm\sqrt{73}$

Q6Sol:

$$\overrightarrow{AB} = (1 - 2)\hat{i} + (-3 + 1)\hat{j} + (-5 - 1)\hat{k} = -i\hat{} - 2\hat{j} - 6\hat{k}$$
$$\overrightarrow{BC} = (3 - 1)\hat{i} + (-4 + 3)\hat{j} + (-4 + 5)\hat{k} = 2i\hat{} - \hat{j} + \hat{k}$$
and $\overrightarrow{CA} = (2 - 3)\hat{i} + (-1 + 4)\hat{j} + (1 + 4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$$\left|\overrightarrow{AB}\right|^2 = 41 = 6 + 35 = \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CA}\right|^2$$

Hence, the triangle is a right angled triangle.

Q7 Sol: The vector which is ⊥ to both $\vec{a}$ & $\vec{b}$ must be ∥ to $\vec{a}X\vec{b}$

Now, $\vec{a}X\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix} = \hat{i}(28 + 4) - \hat{j}(7 - 6) + \hat{k}(-2 - 12)$

$$= 32\hat{\imath} - \hat{\jmath} - 14\hat{k}$$

$$Let\vec{d} = \lambda(\vec{a}X\vec{b}\,) = \lambda(32\hat{\imath} - \hat{\jmath} - 14\hat{k})$$

Then $\vec{c}\,.\,\vec{d} = 15$

$$\Rightarrow 2(32\lambda) + (-1)(-\lambda) + 4(-14\lambda) = 18$$

$\Rightarrow 64\lambda + \lambda - 56\lambda = 18 \Rightarrow 9\lambda = 18$

$$\therefore \lambda = 2$$

$$\because \vec{d} = 2(32\hat{\imath} - \hat{\jmath} - 14\hat{k}) = 64\hat{\imath} - 2\hat{\jmath} - 28\hat{k}$$

Q8 Sol: We have, $|\vec{a}| = |\vec{b}|$=$|\vec{c}| = \lambda$, $\vec{a}.\vec{b}$=$b.\vec{c}$=$\vec{c}.\vec{a} = 0$

$|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$

$$=\vec{a}.\vec{a} + \vec{a}.(\vec{b} + \vec{c}) + \vec{b}.\vec{b} + \vec{b}.(\vec{a} + \vec{c}) + \vec{c}.\vec{c} + \vec{c}.(\vec{a} + \vec{b})$$

$$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$$

$= \lambda^2 + \lambda^2 + \lambda^2$ =$3\lambda^2$

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$$

Let α, β, γ is the angle between $\vec{a} + \vec{b} + \vec{c}$ and $\vec{a}, \vec{b}$ and $\vec{c}$

$$\therefore\ cos\,\alpha = \frac{(\vec{a} + \vec{b} + \vec{c}).\vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$$

$$= \frac{\vec{a}.\vec{a} + \vec{b}.\vec{a} + \vec{c}.a}{\sqrt{3}\lambda.\lambda} = \frac{|\vec{a}|^2}{\sqrt{3}\lambda^2} = \frac{\lambda^2}{\sqrt{3}\lambda^2} = \frac{1}{\sqrt{3}}$$

lly, $cos\,\beta = \frac{1}{\sqrt{3}}, cos\,\gamma = \frac{1}{\sqrt{3}}$

Hence, $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vector $\vec{a}, \vec{b}$ and $\vec{c}$

Q9 Sol:$|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$

$$=\vec{a}.\vec{a} + \vec{a}.(\vec{b} + \vec{c}) + \vec{b}.\vec{b} + \vec{b}.(\vec{a} + \vec{c}) + \vec{c}.\vec{c} + \vec{c}.(\vec{a} + \vec{b})$$

$$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$$

$$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = 3^2 + 4^2 + 5^2 = 9 + 16 + 25 = 50$$

$$\Rightarrow (\vec{a} + \vec{b} + \vec{c})^2 = 50 \Rightarrow a + b + c = \sqrt{50} \Longrightarrow 5\sqrt{2}$$

(iv) Projection of a vector

LEVEL – I

Q1 Sol: Projection of$\vec{a}$ on $\vec{b}$if

$\Rightarrow \vec{a}.\vec{b} = 8$

$\& \vec{b} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k} \quad \Rightarrow |\vec{b}| = \sqrt{4+36+9} = 7$

$= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{8}{7}$

Q2 Sol: Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{1-1}{2} = 0$

Q3 Sol: Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$|\vec{a}| = \sqrt{14}, |\vec{b}| = \sqrt{14}$

$\Rightarrow \vec{a}.\vec{b} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right).(3\hat{i} - 2\hat{j} + \hat{k}) = 10$

$\Rightarrow cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}$

$\Rightarrow \theta = cos^{-1}\frac{5}{7}$

Q4 Sol: Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$, $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{7-3+56}{\sqrt{114}} = \frac{60}{\sqrt{114}}$

LEVEL – II

Q1 Sol: Given $A(0,-1,-2)$, $B(3,1,4)$, $C(5,7,1)$

$\overrightarrow{AB} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$; $\overrightarrow{BC} = 2\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$; $\overrightarrow{AC} = 5\hat{\imath} + 8\hat{\jmath} + 3\hat{k}$

$|\overrightarrow{AB}| = \sqrt{9+4+36} = \sqrt{49} = 7$; $|\overrightarrow{BC}| = \sqrt{4+36+9} = 7$;

$|\overrightarrow{AC}| = \sqrt{25+64+9} = \sqrt{98}$

$AB^2 = 49$; $BC^2 = 49$; $AC^2 = 98$

$AB^2 + BC^2 = AC^2$

$\therefore$ ABC is a right angled triangle rt. angled at B

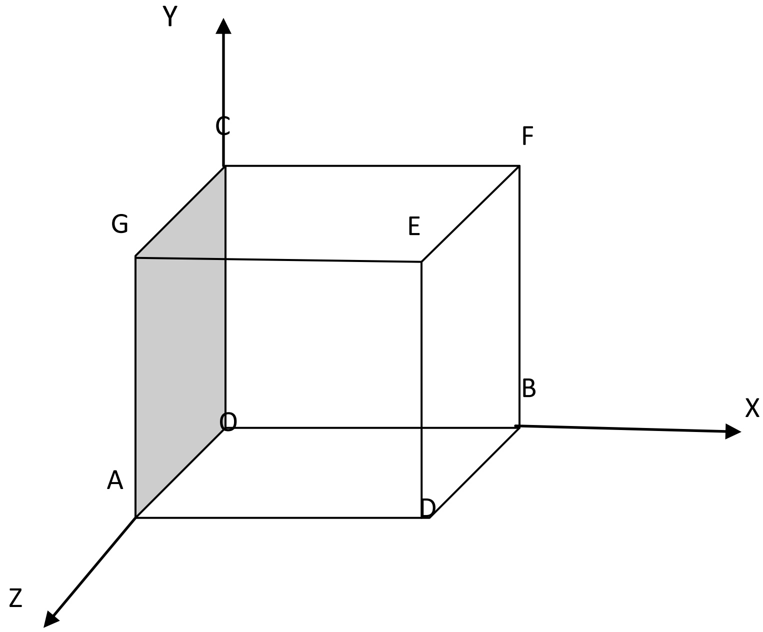
$$cos\,C = \frac{\overrightarrow{BC}.\overrightarrow{AC}}{|\overrightarrow{BC}||\overrightarrow{AC}|} = \frac{49}{7\sqrt{98}} = \frac{1}{\sqrt{2}}$$

$C = 45^0$

$$cos\,A = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{49}{7\sqrt{98}} = \frac{1}{\sqrt{2}}$$

$A = 45^0$

Q2 Sol: Consider a cube of edge =1unit



Then co − ordinates of vertices are A(1,0,0), B(01, ,0), C(0,0,1) ,

D(1,1,0), E(1,1,1), F(0,1,1), G(1,0,1)

Let$\overrightarrow{AF}$and$\overrightarrow{DE}$ are two diagonals

Then $\overrightarrow{CD} = \hat{\imath} + \hat{\jmath} - \hat{k}or\overrightarrow{DC} = -\hat{\imath} - \hat{\jmath} + \hat{k}$

$$\overrightarrow{AF} = -\hat{\imath} + \hat{\jmath} + \hat{k}$$

Let θ be the angle between $\overrightarrow{CD}$&$\overrightarrow{AF}$

Then$\overrightarrow{CD} . \overrightarrow{AF} = |\overrightarrow{CD}|X|\overrightarrow{AF}|Xcos\theta$

= 1-1+1=$\sqrt{3}.\sqrt{3}cos\theta = 1 = 3cos\theta$

$$\Rightarrow cos\theta = \frac{1}{3} \Rightarrow \theta = cos^{-1}\frac{1}{3}$$

Q3Sol:Let x, y, z be real numbers such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

$$\Longrightarrow x(\vec{a} - 2\vec{b} + 3\vec{c}) + y(-3\vec{b} + 5\vec{c}) + z(-2\vec{a} + 3\vec{b} - 4\vec{c}) = 0$$

$$\Rightarrow \vec{a}(x-2z)+\vec{b}(-2x-3y+3z)+\vec{c}(3x+5y-4z)=0$$

$$\Longrightarrow x-2z=0 \dots\dots\dots(1)$$

$$\Longrightarrow -2x-3y+3z=0 \dots\dots\dots.(2)$$

$$\Longrightarrow 3x+5y-4z=0 \dots\dots\dots.(3)$$

$$\therefore (1) \Longrightarrow x=2z$$

$$\therefore \text{ from } (1)\&(2) \Longrightarrow 3y+z=0 \dots\dots.(4)$$

$$\therefore \text{ from } (2)\&(3) \Longrightarrow 5y+2z=0 \dots\dots.(4)$$

$$\text{Solving } (4)\&(5), \text{we get } y=0, z=0$$

$$\text{from}(1) \Longrightarrow x=0$$

Thus, $x=0, y=0, z=0$

hence, $\vec{a}-2\vec{b}+3\vec{c}$, $-3\vec{b}+5\vec{c}$ and $-2\vec{a}+3\vec{b}-4\vec{c}$ are also coplanar

LEVEL – III

Q1 Sol:Let$\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$

$$\vec{a}.\hat{\imath}=cos\frac{\pi}{4}\Longrightarrow a_1=\frac{1}{\sqrt{2}} \quad ;\vec{a}.\hat{\jmath}=cos\frac{\pi}{3}\Longrightarrow a_2=\frac{1}{2}$$

Now ,$\vec{a}.\hat{k}=cos\theta \Longrightarrow a_3=cos\theta$

$$|\vec{a}|=1=\sqrt{\frac{1}{2}+\frac{1}{4}+cos^2\theta}=1$$

$$=\sqrt{\frac{3}{4}+cos^2\theta}=1 \Rightarrow \frac{3}{4}+cos^2\theta=1$$

$$=cos^2\theta=\frac{1}{4}=cos\theta=\frac{1}{2}\Rightarrow \theta=\frac{\pi}{3}$$

Here are the components ,$\vec{a}=\frac{1}{\sqrt{2}}\hat{\imath}+\frac{1}{2}\hat{\jmath}+\frac{1}{2}\hat{k}\&\theta=\frac{\pi}{3}$

Q2 Sol: We have, $|\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$, $\vec{a}.\vec{b}=b.\vec{c}=\vec{c}.\vec{a}=0$

$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$$=\vec{a}.\vec{a}+\vec{a}.(\vec{b}+\vec{c})+\vec{b}.\vec{b}+\vec{b}.(\vec{a}+\vec{c})+\vec{c}.\vec{c}+\vec{c}.(\vec{a}+\vec{b})$$

$$\Rightarrow \left|\vec{a}+\vec{b}+\vec{c}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 0 + 0 + 0$$

$= \lambda^2 + \lambda^2 + \lambda^2 = 3\lambda^2$

$$\left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{3}\lambda$$

Let α, β, γ is the angle between $\vec{a}+\vec{b}+\vec{c}$ and $\vec{a}, \vec{b}$ and $\vec{c}$

$$\therefore \cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right||\vec{a}|}$$

$$= \frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.a}{\sqrt{3}\lambda.\lambda} = \frac{|\vec{a}|^2}{\sqrt{3}\lambda^2} = \frac{\lambda^2}{\sqrt{3}\lambda^2} = \frac{1}{\sqrt{3}}$$

lly, $\cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vector $\vec{a}, \vec{b}$ and $\vec{c}$

Q3 Sol: $\overrightarrow{\beta_1} \parallel \vec{\alpha} \Rightarrow \overrightarrow{\beta_1} = \lambda\vec{\alpha}$

$$\therefore \beta_1 = \lambda(3\hat{\imath} - \hat{\jmath})$$

$$\beta_2 = \beta - \beta_1 = (2-3\lambda)\hat{\imath} + (1+\lambda)\hat{\jmath} - 3\hat{k}$$

$$\beta_2 \perp \alpha \Rightarrow \alpha.\beta_2 = 0$$

$$\Rightarrow 3(2-3\lambda) - (1+\lambda) = 0$$

$$\Rightarrow 6 - 9\lambda - 1 - \lambda = 0 \Rightarrow 5 - 10\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$$

$$\therefore \beta_1 = \frac{1}{2}(3\hat{\imath} - \hat{\jmath})$$

$$\Rightarrow \beta_2 = \beta - \beta_1 = \left(2 - 3\frac{1}{2}\right)\hat{\imath} + \left(1 + \frac{1}{2}\right)\hat{\jmath} - 3\hat{k}$$

$= \frac{1}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} - 3\hat{k}$

Q4 Sol: $Given\ \vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\overrightarrow{AB} = \vec{b} - \vec{a} = -\hat{\imath} + 3\hat{\jmath} + 5\hat{k}; \overrightarrow{BC} = \vec{c} - \vec{a} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$;

$$\overrightarrow{AC} = \vec{a} - \vec{c} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$$

$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}$; $\left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{41}$;

$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$$AB^2 = 35;\ BC^2 = 41;\ AC^2 = 6$$

$$AB^2 + AC^2 = BC^2$$

$\therefore$ ABC is a right angled triangle rt. angled at A

Q5 Sol: $\left|\vec{a} - \vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$

$=1+1-2\cos\theta = 2-2\cos\theta = 2(1-\cos\theta)$

$=2.2 sin^2\frac{\theta}{2} = 4\,sin^2\frac{\theta}{2}$

$$\frac{1}{4}\left|\vec{a}-\vec{b}\right|^2 = sin^2\frac{\theta}{2} \Rightarrow sin\frac{\theta}{2} = \frac{1}{2}\left|\vec{a}-\vec{b}\right|$$

Also, $\frac{1}{2}\left|\vec{a}+\vec{b}\right| = cos\frac{\theta}{2}$

$$\therefore\ tan\frac{\theta}{2} = \frac{\frac{1}{2}\left|\vec{a}-\vec{b}\right|}{\frac{1}{2}\left|\vec{a}+\vec{b}\right|} = \frac{\left|\vec{a}-\vec{b}\right|}{\left|\vec{a}+\vec{b}\right|}$$

(vii)Cross product of two vectors

LEVEL – I

Q1 Sol: If $|\vec{a}| = 3, \left|\vec{b}\right| = 5$; $\vec{a}X\vec{b} = 9$

$$\left|\vec{a}X\vec{b}\right| = ?$$

$$\left|\vec{a}X\vec{b}\right|^2 = |\vec{a}|^2\left|\vec{b}\right|^2 - \left(\vec{a}X\vec{b}\right)^2$$

$=3^2.5^2 - 9^2 = 9X25 - 81 = 225 - 81 = 144$

$$\left|\vec{a}X\vec{b}\right|^2 = 144 \Rightarrow \left|\vec{a}X\vec{b}\right| = \sqrt{144} = 12$$

Q2 Sol: $\vec{a}x\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$

$$= (-14+14)\hat{\imath} - (2-21)\hat{\jmath} + (-2+21)\hat{k}$$
$$= 0.\hat{\imath} + 19\hat{\jmath} + 19\hat{k}$$

$\therefore \left|\vec{a}x\vec{b}\right| = 19\sqrt{2}$

Q3 Sol: : Given $(\vec{x}-\vec{p}).(\vec{x}+\vec{p}) = 80$

$$\Rightarrow \vec{x}^2 - \vec{p}^2 = 80 \Rightarrow |\vec{x}|^2 - |\vec{p}|^2 = 80$$

$$\Rightarrow |\vec{x}|^2 - 1 = 80 \Rightarrow |\vec{x}|^2 = 81 \Rightarrow |\vec{x}| = 9$$

Q4 Sol: Given $\vec{a} = 2\hat{\imath} + 6\hat{\jmath} + 27\hat{k}$; $\vec{b} = \hat{\imath} + 3\hat{\jmath} + p\hat{k}$

$$(\vec{a}x\vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 6 & 27 \\ 1 & 3 & p \end{vmatrix} = \hat{\imath}(6p-81) - \hat{\jmath}(2p-27) + \hat{k}(6-6) = \vec{0}$$

$= \hat{\imath}(6p - 81) - \hat{\jmath}(2p - 27) + \hat{k}(0) = 0$

$= 6p - 81 = 0 \implies p = \frac{27}{2}$

LEVEL – II

Q1 Sol: Given $\vec{a} = 2\hat{\imath} + 6\hat{\jmath} + 14\hat{k}$; $\vec{b} = \hat{\imath} - \lambda\hat{\jmath} + 7\hat{k}$

$$(\vec{a}x\vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix} = \hat{\imath}(42 + 14\lambda) - \hat{\jmath}(14 - 14) + \hat{k}(-2\lambda - 6) = \vec{0}$$

$= \hat{\imath}(42 + 14\lambda) - \hat{\jmath}(14 - 14) + \hat{k}(-2\lambda - 6) = 0$

$= \lambda = -3$

Q2 Sol: $(\vec{a} - \vec{b})x\overrightarrow{(a} + \vec{b}) = \vec{a}x\vec{a} - \vec{b}x\vec{a} + \vec{a}x\vec{b} - \vec{b}x\vec{b}$

$$= 0 + \vec{a}x\vec{b} + \vec{a}x\vec{b} + 0$$

$$= 2(\vec{a}x\vec{b})$$

Q3 Sol: Let θ be the angle between $\vec{a}$&$\vec{b}$

Then, $sin\theta = \frac{|\vec{a}X\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{6}{3*4} = \frac{1}{2}$

$$\theta = sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Q4 Sol: $\vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$

$$\vec{a} \perp \vec{b} \& \vec{a} \perp \vec{c}$$

$$\vec{a} \parallel \vec{b}.\vec{c}$$

$$\vec{a} = \lambda(\vec{b} \times \vec{c}) \implies |\vec{a}| = |\lambda(\vec{b} \times \vec{c})|$$

$$\implies 1 = \pm\lambda|\vec{b}||\vec{c}|sin\frac{\pi}{6}$$

$$\implies 1 = \pm\lambda\frac{1}{2} \implies \lambda = \pm 2$$

$$\vec{a} = \lambda(\vec{b} \times \vec{c}) \quad \implies \quad \vec{a} = \pm 2(\vec{a} \times \vec{b})$$

LEVEL – III

Q1 Sol: $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + \hat{\jmath}.(\hat{\imath} \times \hat{k}) + \hat{k}.(\hat{\imath} \times \hat{\jmath})$

$= \hat{\imath}.\hat{\imath} + \hat{\jmath}.(-\hat{\jmath}) + \hat{k}.\hat{k}$ =1 – 1+1=1

Q2 Sol: Vector $\vec{a}$&$\vec{b}$ are such that$|\vec{a}| = \sqrt{3}$; $|\vec{b}| = \frac{2}{3}$

&$|\vec{a}X\vec{b}| = 1$

Let θ is angle between $\vec{a}$&$\vec{b}$

Then $sin\theta = \frac{|\vec{a}X\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{3}.\frac{2}{3}} = \frac{\sqrt{3}}{2}$

$$\therefore \ \theta = sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Q3 Sol:

: $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$; $\vec{b} = \hat{\jmath} - \hat{k}$; $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$

$$(\vec{a})X(\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \hat{\imath}(c_3 - c_2) - \hat{\jmath}(c_3 - c_1) + \hat{k}(c_2 - c_1) = \hat{\jmath} - \hat{k}$$

$$\Longrightarrow \ c_3 - c_2 = 0; \ c_3 - c_1 = -1 \ldots\ldots\ldots (i); \ c_2 - c_1 = -1 \ldots\ldots\ldots\ldots\ldots (ii)$$

$\Longrightarrow c_3 = c_2$

$$(\vec{a}).(\vec{c}) = 3 \ \Longrightarrow c_1 + c_2 + c_3 = 3$$

$$\Longrightarrow c_1 + 2c_3 = 3 \ldots\ldots\ldots\ldots\ldots\ldots..(iii)$$

Solving (i) & (ii) we get,

$\Longrightarrow c_3 = \frac{2}{3}$

$$\Longrightarrow c_2 = c_3.1 = \frac{2}{3}.1$$

$\Longrightarrow c_1 = \frac{5}{3}$

$\Longrightarrow \vec{c} = \frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$

Q4 Sol:$\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$&$\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

$$(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = \vec{c} \times \vec{d} - \vec{b} \times \vec{d}$$

$$\Rightarrow \vec{a}(\vec{b} - \vec{c}) = \vec{d}(\vec{c} - \vec{b})$$

$$\Rightarrow \vec{a}(\vec{b} - \vec{c}) - \vec{d}(\vec{c} - \vec{b}) = 0$$

$$\Rightarrow \vec{a}(\vec{b} - \vec{c}) - \vec{d}(\vec{b} - \vec{c}) = 0$$

$$\Rightarrow (\vec{a}-\vec{d}).(\vec{b}-\vec{c})=0$$

Hence, $(\vec{a}-\vec{d}) \parallel (\vec{b}-\vec{c})$

Q5 Sol:let$\overrightarrow{\beta_1}=2\hat{\imath}-\hat{\jmath}+3\hat{k}$ and $\vec{\alpha}=2\hat{\imath}+4\hat{\jmath}-2\hat{k}, \overrightarrow{\beta_1} \parallel \vec{\alpha} \Rightarrow \overrightarrow{\beta_1}=\lambda\vec{\alpha}$

$$\therefore \beta_1=\lambda(2\hat{\imath}+4\hat{\jmath}-2\hat{k})$$

$$\beta_2=\beta-\beta_1=2(1-\lambda)\hat{\imath}-(1+4\lambda)\hat{\jmath}+(3+2\lambda)\hat{k}$$

$$\beta_2 \perp \alpha \Rightarrow \alpha.\beta_2=0$$

$$\Rightarrow 4(1-\lambda)-4(1+4\lambda)-2(3+2\lambda)=0$$

$$\Rightarrow -24\lambda-6=0 \Longrightarrow \lambda=-\frac{1}{4}$$

$\therefore\ \beta_1=-\frac{1}{4}(2\hat{\imath}+4\hat{\jmath}-2\hat{k}) =-\frac{1}{2}\hat{\imath}-\hat{\jmath}+\frac{1}{2}\hat{k}$

$$\Rightarrow \beta_2=\ \beta-\beta_1=2\left(1+\frac{1}{4}\right)\hat{\imath}-\left(1+4(-\frac{1}{4})\right)\hat{\jmath}+(3+2(-\frac{1}{4})\hat{k} \qquad = \quad \frac{5}{2}(\hat{\imath}+\hat{\jmath})$$

$$2\hat{\imath}-\hat{\jmath}+3\hat{k}=\left(-\frac{1}{2}\hat{\imath}-\hat{\jmath}+\frac{1}{2}\hat{k}\right)+\frac{5}{2}(\hat{\imath}+\hat{\jmath})$$

(viii)Area of a triangle & Area of a parallelogram

LEVEL – I

Q1 Sol: Given Sides = $\overrightarrow{a_1}=3\hat{\imath}+\hat{\jmath}-2\hat{k}$; $\overrightarrow{a_2}=\hat{\imath}-3\hat{\jmath}+4\hat{k}$

$$\overrightarrow{a_1}X\overrightarrow{a_2}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=\hat{\imath}(4-6)-\hat{\jmath}(12+2)+\hat{k}(-9-1)$$

$$=\ -2\hat{\imath}-14\hat{\jmath}-10\hat{k}$$

Area of $\parallel gm\ =|\overrightarrow{a_1}X\overrightarrow{a_2}|=\left|-2\hat{\imath}-14\hat{\jmath}-10\hat{k}\right|$

$=\ \sqrt{4+196+100}=\sqrt{300}=(10\sqrt{3})=10\sqrt{3}\ \ sq\ \ units$

Q2 Sol: area of llgm =$\left|\vec{a}x\vec{b}\right|$

Q3 Sol:

Given vertices of ΔABC , A(1,1,1), B(1,2,3), C(2,3,1)

$\overrightarrow{AB}=0\hat{\imath}+\hat{\jmath}+2\hat{k}$&$\overrightarrow{AC}=\hat{\imath}+\widehat{2\jmath}+0\hat{k}$

$$\therefore \overrightarrow{AB}X\overrightarrow{AC}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=\hat{\imath}(-4)-\hat{\jmath}(-2)+\hat{k}(0-1)$$

$$=-4\hat{\imath}+2\hat{\jmath}-\hat{k}$$

$$Areaof\ \Delta = \frac{1}{2}\left|\overrightarrow{AB}X\overrightarrow{AC}\right| = \frac{1}{2}\left||-4\hat{\imath} + 2\hat{\jmath}) - \hat{k}\right|$$

$Areaof\ \Delta = \frac{1}{2}\sqrt{16+4+1} = \frac{\sqrt{21}}{2}$ Sq units

LEVEL – II

Q1 Sol: Given diagonal = $\overrightarrow{d_1} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$; $\overrightarrow{d_2} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$

$$\overrightarrow{d_1}X\overrightarrow{d_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{\imath}(4-6) - \hat{\jmath}(12+2) + \hat{k}(-9-1)$$

$$= -2\hat{\imath} - 14\hat{\jmath} - 10\hat{k}$$

Area of ∥ $gm = \frac{1}{2}\left|\overrightarrow{d_1}.\ \overrightarrow{d_2}\right| = \frac{1}{2}\left|-2\hat{\imath} - 14\hat{\jmath} - 10\hat{k}\right|$

$$= \frac{1}{2}\sqrt{4+196+100} = \frac{1}{2}\sqrt{300} = \frac{1}{2}\left(10\sqrt{3}\right) = 5\sqrt{3}\ sq\ \ units$$

Q2 Sol: Let $\vec{a}, \vec{b}, \vec{c}$ are the position vector of vertices of ΔABC

$$\overrightarrow{AB} = \vec{b} - \vec{a} \& \overrightarrow{BC} = \vec{c} - \vec{b}$$

Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}X\overrightarrow{BC}\right| = \frac{1}{2}\left|\left(\vec{b} - \vec{a}\right)X\left(\vec{c} - \vec{b}\right)\right|$

$=\frac{1}{2}\left|\vec{b}X\vec{c} - \vec{b}X\vec{b} - \vec{a}X\vec{c} + \vec{a}X\vec{b}\right|$

$=\frac{1}{2}\left|\vec{b}X\vec{c} - \vec{a}X\vec{c} + \vec{a}X\vec{b}\right|$

$$= \frac{1}{2}\left|\vec{b}X\vec{c} + \vec{a}X\vec{c} + \vec{a}X\vec{b}\right|$$

Q3 Sol:

Given vertices of ΔABC , A(1,1,2), B(2,3,5), C(1,5,5)

$\overrightarrow{AB} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} \& \overrightarrow{AC} = 0\hat{\imath} + \hat{4\jmath} + 3\hat{k}$

$$\therefore \overrightarrow{AB}X\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = \hat{\imath}(6-12) - \hat{\jmath}(3-0) + \hat{k}(4-0)$$

$$= -6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

$$Area\ of\ \Delta = \frac{1}{2}\left|\overrightarrow{AB}X\overrightarrow{AC}\right| = \frac{1}{2}\left|-6\hat{\imath} - 3\hat{\jmath} + 4\hat{k}\right|$$

$Areaof\ \Delta = \frac{1}{2}\sqrt{36+9+16} = \frac{\sqrt{61}}{2}$ Sq units

TOPIC-10

THREE DIMENSIONAL GEOMETRY

(i)Direction Ratios and Direction Cosines

LEVEL-I

Q1 Sol: Dr's of line joining (1,0,0) and (0,1,1) is $-\hat{\imath}+\hat{\jmath}+\hat{k}$

Also, $\left|-\hat{\imath}+\hat{\jmath}+\hat{k}\right| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{3}$

$\therefore$ Dc′s of line joining (1,0,0) and (0,1,1)are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Q2 Sol: Direction cosines of the line passing through the pts $(-2,4,-5), (1,2,3)$ are

$$l = \frac{3}{\sqrt{3^2+2^2+8^2}} = \frac{3}{\sqrt{9+4+64}} = \frac{3}{\sqrt{77}}$$

$$m = \frac{-2}{\sqrt{3^2+2^2+8^2}} = \frac{-2}{\sqrt{77}} \quad , n = \frac{8}{\sqrt{3^2+2^2+8^2}} = \frac{8}{\sqrt{77}}$$

Q3 Sol: Let line is making θ with each axis

$$l = \cos\theta = m = n = k(say)$$

$$\therefore\ l^2+m^2+n^2 = 1 \Longrightarrow k^2+k^2+k^2 = 1 \Longrightarrow 3k^2 = 1$$

$$\Longrightarrow k^2 = -\frac{1}{3} \Longrightarrow k = \pm\frac{1}{\sqrt{3}} \therefore l = \pm\frac{1}{\sqrt{3}} = m = n$$

LEVEL-II

Q1 Sol: Given line is $\frac{x-3}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$

$\therefore$ Direction Cosines of a line $\parallel$ to the given line are

$$l = \frac{-3}{\sqrt{(-3)^2+(-2)^2+6^2}} = \frac{-3}{\sqrt{9+4+36}} = \frac{-3}{7}$$

$$m = \frac{-2}{7} \quad , \quad n = \frac{6}{7}$$

$\therefore$ Direction Cosines are $\frac{-3}{7}, \frac{-2}{7}, \frac{6}{7}$

Q2 Sol: Given line is $\frac{x-5}{-3} = \frac{y+7}{-2} = \frac{z+2}{6}$

∴ Direction ratio of aline ∥ to the given line are , $< -3, \quad -2, \quad 6 >$

Q3 Sol: We have, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+6}{3}$

∴ Direction Cosines of a line ∥ to the given line are

$l = \frac{2}{\sqrt{(2)^2+(-1)^2+3^2}} = \frac{-1}{\sqrt{4+1+9}} = \frac{2}{\sqrt{14}}$

$m = \frac{-1}{\sqrt{14}} \quad , \quad n = \frac{3}{\sqrt{14}}$

∴ Direction Cosines are $\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

*Q*4 *Sol*:Let line is making θ with each axis

$$l = \cos\theta = m = n = k(say)$$

$$\therefore \ l^2 + m^2 + n^2 = 1 \Rightarrow k^2 + k^2 + k^2 = 1 \Rightarrow 3k^2 = 1$$

$$\Rightarrow k^2 = -\frac{1}{3} \Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

$$\therefore l = \pm\frac{1}{\sqrt{3}} = m = n$$

(ii) Cartesian and Vector equation of a line in space & conversion of one into another form

Q1 Sol: Vector Equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{-2}$ is

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$$

Q2 Sol: Equation of line ∥ to $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ through the pt. (1,2,3) is

$$\frac{x-1}{-3} = \frac{y-2}{2} = \frac{z-3}{6}$$

Q3 Sol: Cartesian form of the line $\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ is

$$\frac{x-1}{2} = \frac{y+2}{1} = \frac{z-1}{2} = \lambda$$

Q4 Sol:Cartesian form of plane is $2x - 3y + z + 4 = 0$

(iii) Co-planer and skew lines

LEVEL-II

Q1 Sol: Here $\vec{a_1} = \hat{\imath} - \hat{\jmath} - \hat{k}, \vec{a_2} = 2\hat{\imath} - \hat{\jmath}$

$$\vec{b_1} = 2\hat{\imath} + \hat{\jmath}, \vec{b_2} = \hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\therefore (\vec{a_2} - \vec{a_1}) = \hat{\imath} + 0\hat{\jmath} + \hat{k}$$

$Now, (\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1}) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 1(-1-0) - 0(-2-0) + 1(2-1) = -1+1 = 0$

∴ The lines intersect each other

The Cartesian form of lines are

$$\frac{x-1}{2} = \frac{y+1}{1} = \frac{z+1}{0} = \lambda \ldots\ldots\ldots\ldots\ldots..(i)$$

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-0}{-1} = \mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(ii)$$

∴ any pt. on line (i) is

$$x = 2\lambda + 1, y = \lambda - 1, z = -1$$

∴ any pt. on line (ii) is

$$x = \mu + 2, y = \mu - 1, z = -\mu$$

∵ the pts on each line common

$\mu = 1, \mu - 1 = \lambda - 1 \Rightarrow \lambda = 1$

∴ the pt of intersection is $(3,0,-1)$

Q2 Sol: Let A, B, C are the dR's of normal to the plane containing the pts

(0,-1,-1),(4,51) & (3,9,4)

∴ $A(x-0) + B(y+1) + C(z+1) = 0 \ldots\ldots..$ (i)is req. plane

Hence $A(4-0) + B(5+1) + C(1+1) = 0$

$\Rightarrow 4A + 6B + 2C = 0 \ldots\ldots\ldots\ldots..$(ii)

& $3A + 10B + 5C = 0 \ldots\ldots..$(iii)

$$\therefore \frac{A}{30-20} = \frac{-B}{20-6} = \frac{C}{40-18}$$

$$\Rightarrow \frac{A}{10} = \frac{B}{-14} = \frac{C}{22} \Rightarrow \frac{A}{5} = \frac{B}{-7} = \frac{C}{11}$$

∴ the plane (i) is

$5x + (-7)(y+1) + 11(z+1) = 0 \Rightarrow 5x - 7y + 11z + 4 = 0$

If the pt. $(-4,4,4)$ is on the plane

i. e all the given four pts are coplaner

Q3 Sol: Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda \ldots\ldots\ldots\ldots\ldots..(1)$

and $\frac{x-4}{5}=\frac{y-1}{2}=z=\mu \ldots\ldots\ldots...(2)$

$(1) \Longrightarrow x=1+2\lambda;\ \ y=2+3\lambda;\ \ z=3+4\lambda$

$(2) \Longrightarrow x=4+5\mu;\ \ y=1+2\mu;\ \ z=\mu$

line (1)& (2) intersect, then points of intersection concides

$\therefore 1+2\lambda=4+5\mu\ \ ,\ 2+3\lambda=1+2\mu\ \ , 3+4\lambda=\mu$

$\Longrightarrow 2\lambda-5\mu=3 \ldots\ldots\ldots\ldots(3)$

$3\lambda-2\mu=-1 \ldots\ldots\ldots\ldots.(4)$

$4\lambda-\mu=-3 \ldots\ldots\ldots\ldots\ldots(5)$

Solving (3)&(4), we get

$\lambda=-1, \mu=-1$

This value satisfies eq(5)

$\Longrightarrow$ line (1) & (2) intersect

$\therefore$ point of intersection is $(-1,-1,-1)$

LEVEL-III

Q1 Sol: Coplaner condition for two lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \& \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$$

$$=\begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=2(5-10)-1(-15+5)=-10+10=0$$

$\therefore$ The lines are coplaner

The eq of a plane is $=\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}=0$

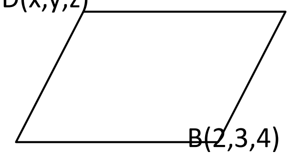
$\therefore (x+3)(5-10)-(y-1)(-15+5)+(z-5)(-6+1)=0$

$\therefore\ -5(x+3)-(y-1)(-10)+(z-5)(-5)=0$

$\therefore -5x+10y-5z-15-10+25=0$

$\therefore -5x+10y-5z=0 \quad \text{or} -x+2y-z=0$

Q2 Sol: Vector equation of side AB is



$\vec{r}=\left(4\hat{\imath}+5\hat{\jmath}+10\hat{k}\right)+\lambda\left(-2\hat{\imath}-2\hat{\jmath}-6\hat{k}\right)$

& vector equation of side BC is

$\vec{r}=\left(2\hat{\imath}+3\hat{\jmath}+4\hat{k}\right)+\mu\left(-\hat{\imath}-\hat{\jmath}-5\hat{k}\right)$

At co-ordinates of D are (x, y, z)

$\therefore\ \frac{x+2}{2}=\frac{5}{2},\frac{y+3}{2}=\frac{7}{2},\frac{z+4}{2}=\frac{9}{2}$

$\therefore$ D(3,4,5)

Q3 Sol: Let a, b, c be the dR's of line passing through the pt(1,1,1)

$$\therefore \text{line}\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c}\ \dots\dots\dots\dots\dots\dots (\mathbf{1})$$

$\because$ line(1) & $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ intersect

$\therefore (\overrightarrow{a_2}-\overrightarrow{a_1}).\left(\overrightarrow{b_2}X\overrightarrow{b_1}\right)=0$

$\therefore \begin{vmatrix}0&1&2\\a&b&c\\2&3&4\end{vmatrix}=0 \Rightarrow -1(4a-2c)+2(3a-2b)=0$

$\Rightarrow 2a-4b+2c=0 \Rightarrow a-2b+c=0 \dots\dots\dots..(2)$

Alsoline(1)& $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ intersect

$\therefore (\overrightarrow{a_2}-\overrightarrow{a_1}).\left(\overrightarrow{b_2}X\overrightarrow{b_1}\right)=0$

$\therefore \begin{vmatrix}-3&2&-2\\a&b&c\\1&2&4\end{vmatrix}=0 \Rightarrow -3(4b-2c)-2(4a-c)-2(2a-b)=0$

$=-12a\text{-}10b\text{+}8c=0 \quad \Rightarrow 6a+5b-4c=0 \dots\dots\dots\dots\dots (3)$

From (2) & (3)

$\frac{a}{8-10}=\frac{-b}{-4-6}=\frac{c}{5+12} \Rightarrow \frac{a}{-2}=\frac{b}{10}=\frac{c}{17}$

$\therefore$ Reqlineis

$\frac{x-1}{-2}=\frac{y-1}{10}=\frac{z-1}{17}$

Q4 Sol:Let A, B, C are the dR's of Normal to the plane containing the pts

(0,-1,-1),(4,51) & (3,9,4)

$\therefore A(x-0)+B(y+1)+C(z+1)=0 \ldots\ldots..(i) is\ req.plane$

Hence $\mathrm{A}(4-0)+\mathrm{B}(5+1)+\mathrm{C}(1+1)=0$

$\Longrightarrow 4A+6B+2C=0 \ldots\ldots\ldots\ldots..(ii)$

& $\quad 3A+10B+5C=0 \ldots\ldots..(iii)$

$$\therefore \frac{A}{30-20}=\frac{-B}{20-6}=\frac{C}{40-18}$$

$$\Longrightarrow \frac{A}{10}=\frac{B}{-14}=\frac{C}{22} \Longrightarrow \frac{A}{5}=\frac{B}{-7}=\frac{C}{11}$$

$\therefore$ the plane (i) is

$5\mathrm{x}+(-7)(\mathrm{y}+1)+11(\mathrm{z}+1)=0 \Longrightarrow 5\mathrm{x}-7\mathrm{y}+11\mathrm{z}+4=0$

If the pt. $(-4,4,4)$ is on the plane

i.e all the given four pts are coplaner

(iv) Shortest distance between two lines

LEVEL-II

Q1 Sol: (a)Vector form of line l_1 is $\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$

Vector form of line l_2 is $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$

$\therefore \overrightarrow{a_2}-\overrightarrow{a_1}=\hat{i}-3\hat{j}-2\hat{k}$

$\overrightarrow{b_1}=\hat{i}-\hat{j}+\hat{k}$, $\overrightarrow{b_2}=2\hat{i}+\hat{j}+2\hat{k}$

$\therefore$ S.D between the line l_1&l_2 is

$$d=\left|\frac{(\overrightarrow{a_2}-\overrightarrow{a_1}).(\overrightarrow{b_2}X\overrightarrow{b_1})}{|\overrightarrow{b_2}X\overrightarrow{b_1}|}\right|$$

$$\text{Now, }(\overrightarrow{b_2}X\overrightarrow{b_1})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}=\hat{i}(-2-1)-\hat{j}(2-2)+\hat{k}(1+2)$$

$$=-3\hat{i}-0\hat{j}+3\hat{k}$$

$$\therefore |\overrightarrow{b_2}X\overrightarrow{b_1}| = \sqrt{9+0+9} = 3\sqrt{2}$$

$$d=\left|\frac{(\overrightarrow{a_2}-\overrightarrow{a_1}).(\overrightarrow{b_2}X\overrightarrow{b_1})}{|\overrightarrow{b_2}X\overrightarrow{b_1}|}\right|=\left|\frac{(\hat{i}-3\hat{j}-2\hat{k}).(-3\hat{i}-0\hat{j}+3\hat{k})}{3\sqrt{2}}\right|=\left|\frac{-3+0-6}{3\sqrt{2}}\right|$$

$=\frac{9}{3\sqrt{2}}=\frac{3\sqrt{2}}{2}$units

(b)we have, $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$

$\vec{r}=(4\hat{i}+2\mu)\hat{i}+(5+3\mu)\hat{j}+(6+\mu)\hat{k}$.

$\Longrightarrow \vec{r}=(4\hat{i}+5\hat{j}+6\hat{k})+\lambda(2\hat{i}+3\hat{j}+\hat{k})$

$\overrightarrow{a_1} = \hat{\imath} + \widehat{2\jmath} + 3\hat{k}$;$\overrightarrow{a_2} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$

$\overrightarrow{b_1} = \hat{\imath} - 3\hat{\jmath} + 2\hat{k}$;$\overrightarrow{b_2} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$

$\therefore$ S.D between the line is

$$d = \left|\frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1})}{|\overrightarrow{b_2} X \overrightarrow{b_1}|}\right|$$

$$Now, (\overrightarrow{b_2} X \overrightarrow{b_1}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \hat{\imath}(-3-6) - \hat{\jmath}(1-4) + \hat{k}(3+6)$$

$$= -9\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$$

$$\therefore |\overrightarrow{b_2} X \overrightarrow{b_1}| = \sqrt{(-9)^2 + 3^2 + 9^2} = 3\sqrt{19}$$

$$d = \left|\frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1})}{|\overrightarrow{b_2} X \overrightarrow{b_1}|}\right| = \left|\frac{(3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}).(-9\hat{\imath} + 3\hat{\jmath} + 9\hat{k})}{3\sqrt{19}}\right| = \left|\frac{-27 + 9 + 27}{3\sqrt{19}}\right|$$

$$d = \frac{3}{\sqrt{19}}$$

Q2 Sol: $Let\ \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \ldots\ldots\ldots\ldots\ldots..(1)$

$$and\ \frac{x-4}{5} = \frac{y-1}{2} = z = \mu \ldots\ldots\ldots...(2)$$

(1)$\Longrightarrow$ $x = 1 + 2\lambda;\ y = 2 + 3\lambda;\ z = 3 + 4\lambda$

(2)$\Longrightarrow$ $x = 4 + 5\mu;\ y = 1 + 2\mu;\ z = \mu$

line (1)& (2) intersect, then points of intersection concides

$\therefore 1 + 2\lambda = 4 + 5\mu$, $2 + 3\lambda = 1 + 2\mu$,$3 + 4\lambda = \mu$

$\Longrightarrow 2\lambda - 5\mu = 3 \ldots\ldots\ldots\ldots(3)$

$3\lambda - 2\mu = -1 \ldots\ldots\ldots\ldots.(4)$

$4\lambda - \mu = -3 \ldots\ldots\ldots\ldots\ldots(5)$

Solving (3) & (4), we get

$\lambda = -1, \mu = -1$

This value satisfies eq(5)

$\Longrightarrow$ line (1) &(2)intersect

$\therefore$ point of intersection is $(-1, -1, -1)$

Q3 Sol:we have, $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$, and

$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(4\hat{i} - 2\hat{j} + 2\hat{k})$

$\overrightarrow{a_1} = \hat{\imath} + \hat{\jmath}$;$\overrightarrow{a_2} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$

$\overrightarrow{b_1} = 2\hat{i} - \hat{j} + \hat{k}$;$\overrightarrow{b_2} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = \hat{\imath} - \hat{k}$

$$d = \left| \frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1})}{|\overrightarrow{b_2} X \overrightarrow{b_1}|} \right|$$

$$Now, (\overrightarrow{b_2} X \overrightarrow{b_1}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 4 & -2 & 2 \end{vmatrix} = \hat{\imath}(-2+2) - \hat{\jmath}(4-4) + \hat{k}(-4+4) = 0$$

$\therefore$ S.D between the line is = 0

Q4 Sol: we have, $\vec{r} = (1-t)\hat{\imath} + (t-2)\hat{\jmath} + (3-t)\hat{k}$ and

$\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} + (2s+1)\hat{k}$

$\Longrightarrow \vec{r} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + t(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$

$\Longrightarrow \vec{r} = (\hat{\imath} - \hat{\jmath} - \hat{k}) + t(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$

$\overrightarrow{a_1} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$; $\overrightarrow{a_2} = \hat{\imath} - \hat{\jmath} - \hat{k}$

$\overrightarrow{b_1} = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$; $\overrightarrow{b_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\therefore \overrightarrow{a_2} - \overrightarrow{a_1} = \hat{\jmath} - 4\hat{k}$

$$d = \left| \frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1})}{|\overrightarrow{b_2} X \overrightarrow{b_1}|} \right|$$

$$Now, (\overrightarrow{b_2} X \overrightarrow{b_1}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$$

$$\therefore |\overrightarrow{b_2} X \overrightarrow{b_1}| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{29}$$

$$d = \left| \frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_2} X \overrightarrow{b_1})}{|\overrightarrow{b_2} X \overrightarrow{b_1}|} \right| = \left| \frac{(\hat{\jmath} - 4\hat{k}).(2\hat{\imath} - 4\hat{\jmath} - 3\hat{k})}{\sqrt{29}} \right| = \left| \frac{-4+12}{\sqrt{29}} \right|$$

$$d = \frac{8}{\sqrt{29}}$$

Q5 Sol: Distance between the given ∥ planesis

$$\mathrm{d} = \left| \frac{5-4}{\sqrt{1^2+1^2+1^2}} \right| = \frac{1}{\sqrt{3}} \text{units}$$

Q6 Sol: The eq. of line passing through the pt(3,0,4)& ∥ *to* the given line

$$\frac{x-3}{5} = \frac{y-0}{-2} = \frac{z+4}{4}$$

$\therefore$ vectoreq is $\vec{r} = (3\hat{\imath} - 4\hat{k}) + \lambda(5\hat{\imath} - 2\hat{\jmath} + 4\hat{k})$

Given line is $\vec{r} = (\hat{\imath} + 3\hat{\jmath} - \hat{k}) + \mu(5\hat{\imath} - 2\hat{\jmath} + 4\hat{k})$

Now, $(\overrightarrow{a_2} - \overrightarrow{a_1}) = (-2\hat{\imath} + 3\hat{\jmath} + 3\hat{k})$ & $\vec{b} = 5\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

$$d = \left|\frac{(\overrightarrow{a_2} - \overrightarrow{a_1})X(\vec{b})}{|\vec{b}|}\right|$$

$$\text{Now,}(\overrightarrow{a_2} - \overrightarrow{a_1})X(\vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -2 & 4 \\ 2 & 3 & 3 \end{vmatrix}$$

$$= \hat{\imath}(-6-12) - \hat{\jmath}(15-8) + \hat{k}(15+4) = -18\hat{\imath} - 7\hat{\jmath} + 19\hat{k}$$

$$|(\overrightarrow{a_2} - \overrightarrow{a_1})X(\vec{b})| = \sqrt{(-18)^2 + (-7)^2 + 19^2} = \sqrt{734}$$

$$|\vec{b}| = \sqrt{25+16+4} = \sqrt{45}$$

$$\therefore d = \frac{\sqrt{734}}{\sqrt{45}} = 7.75\ units$$

(v) Cartesian and Vector equation of a plane in space & conversion of one into another form

LEVEL I

Q1 Sol: Let A, B, C be the $dR's$ of Normal to the plane

$\therefore$ plane passing through origin is $Ax + By + Cz = 0$ ……….(i)

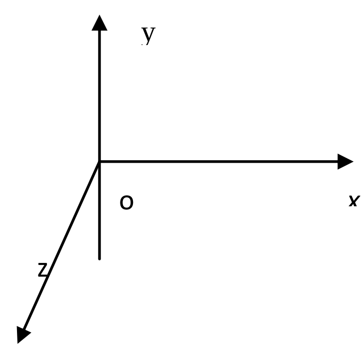
$\because$ the plane (i) is $\perp$ to $X-$axis

$\therefore$ Normal of plane(i) is $\parallel$ to $X-$axis

$\therefore \frac{A}{1} = \frac{B}{0} = \frac{C}{0} = \lambda$

$\therefore A = \lambda, B = 0, C = 0$

$\therefore$ the plane is $\lambda x = 0 \Rightarrow x = 0$



Q2 Sol:Equation of plane with x, y, z intercepts 2,3,4 is

x/2+y/3+z/4=1

Or 6x+4y+3z=12

Q3 Sol: We have,

$\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} - 2\hat{k}) + 1 = 0$

$\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} - 2\hat{k}) = -1$

$\vec{r} \cdot (-1)(6\hat{\imath} - 3\hat{\jmath} - 2\hat{k}) = 1$

$\vec{r} \cdot (-6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) = 1$

Vector normal to the plane is -6,3,2,

Dividing by; $\sqrt{(-6)^2 + 3^2 + 2^2} = 7$

Direction cosines of the unit perpendicular to the plane is $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$

Q4 Sol:(a) We have, $\vec{r} \cdot (\hat{\imath} + \hat{\jmath} - \hat{k}) = 2$ ……….(i)

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i)

$\therefore (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 2$

$\Rightarrow x + y - z = 2$

(b) We have, $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$ ……….(i)

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i)

$\therefore (x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

$\Rightarrow 2x + 3y - 4z = 1$

LEVEL II

Q1 Sol:We have the position vector of point $(5, 2, -4)$ as

$$\vec{a} = 5i\hat{} + 2\hat{}j - 4k\hat{} \text{ andthe}$$

normal vector N perpendicular to the plane as $N = 2i\hat{} + 3\hat{}j - k\hat{}$

Therefore, the vector equation of the plane is given by $(r - a).N = 0$

or$[r - (5i\hat{} + 2\hat{}j - 4k\hat{})] \cdot (2i\hat{} + 3\hat{}j - k\hat{}) = 0$ _ … (1)

Transforming (1)into Cartesian form, wehave

$[(x - 5)i\hat{} + (y - 2)\hat{}j + (z + 4)k\hat{}] \cdot (2i\hat{} + 3\hat{}j - k\hat{}) = 0$

or $2(x - 5) + 3(y - 2) - 1(z + 4) = 0$

$$\text{i.e. } 2x + 3y - z = 20$$

which is the cartesian equation of the plane.

Q2 Sol:normal vector $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

Unit normal vector $\hat{n} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$

$\therefore$ eqn of plane $\vec{r}.\hat{n} = p$

$\Rightarrow \vec{r}.((3i\hat{} + 5j\hat{} - 6k\hat{})/\sqrt{70}) = 7 \Rightarrow \vec{r}.(3i\hat{} + 5j\hat{} - 6k\hat{}) = 7\sqrt{70}$

Q3 Sol:normal vector $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

$\vec{a} = \hat{i} - 2\hat{k}$

equation of plane through $\vec{a}$ and having normal vector is

$\overrightarrow{(r - \vec{a})}.\vec{n} = 0$

$\Rightarrow [\vec{r} - (\hat{\imath} - 2\hat{k})].(\hat{\imath} + \hat{\jmath} - \hat{k}) = 0$ *is a vector eqn.*

Cartesian equation is

$[(x-1)\hat{\imath} + y\hat{\jmath} + (z+2)\hat{k}].(\hat{\imath} + \hat{\jmath} - \hat{k}) = 0$

$\Rightarrow (x-1) + y - (z+2) = 0$

$\Rightarrow x + y - z = 3$

$(\boldsymbol{vi})$ ***AngleBetween*** $(\boldsymbol{i})$***Twolines*** $(\boldsymbol{ii})$***Twoplanes*** $(\boldsymbol{iii})$***Line***&*plane*

LEVEL-I

$Q1\ Sol$: Direction cosines of the line with dR's 1,1,2 are

$$l_1 = \frac{1}{\sqrt{1^2+1^2+2^2}} = \frac{1}{\sqrt{6}}\ ,m_1 = \frac{1}{\sqrt{6}}\ ,n_1 = \frac{2}{\sqrt{6}}$$

Direction cosines of the line with dR's$\sqrt{3}-1, -\sqrt{3}-1, 4$ *are*

$$l_2 = \frac{\sqrt{3}-1}{\sqrt{(\sqrt{3}-1)^2 + (-\sqrt{3}-1)^2 + 4^2}} = \frac{\sqrt{3}-1}{\sqrt{24}}, m_2 = \frac{-\sqrt{3}-1}{\sqrt{24}}, n_2 = \frac{4}{\sqrt{24}}$$

∴ angle (θ) between the lines is $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$$= \frac{1}{\sqrt{6}}.\frac{\sqrt{3}-1}{\sqrt{24}} + \frac{1}{\sqrt{6}}.\frac{-\sqrt{3}-1}{\sqrt{24}} + \frac{2}{\sqrt{6}}.\frac{4}{\sqrt{24}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{6}\sqrt{24}} - \frac{\sqrt{3}+1}{\sqrt{6}\sqrt{24}} + \frac{8}{\sqrt{6}\sqrt{24}} = \frac{\sqrt{3}-1}{12} - \frac{\sqrt{3}+1}{12} + \frac{8}{12}$$

$$= \frac{\sqrt{3}-1-\sqrt{3}-1+8}{12} = \frac{6}{12} = \frac{1}{2}$$

$$\Rightarrow \theta = cos^{-1}\left(\frac{1}{2}\right) = 60^0$$

Q2 Sol: The angle θ between the line & *plane* (*given*)

$$sin\,\theta = \frac{9-4+2}{\sqrt{9+16+1}\sqrt{9+1+4}} = \frac{7}{\sqrt{14}\sqrt{26}} = \frac{7}{2\sqrt{91}}$$

$\therefore \theta = sin^{-1}\left(\frac{7}{2\sqrt{91}}\right)$

Q3 Sol: *The line* $\frac{x-2}{9} = \frac{y-1}{\lambda} = \frac{z+3}{-6}$ *is* $\perp$ *to the plane*

$$\therefore 3x - y - 2z = 7$$

$$\therefore \frac{9}{3} = \frac{\lambda}{-1} = \frac{-6}{-2} \quad \Rightarrow \therefore \lambda = -3$$

Q4 Sol: Given planes are

$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$................(1)

and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$.................(2)

dn's of normal to plane (1) are $< 2, 2, -3 >$ *and* dn's of any normal to plane(2) are $< 3, -3, 5 >$

Let θ is the angle between two plane (1) & (2), then

$$\cos\theta = \frac{|2x3 + 2(-3) + (-3)x5|}{\sqrt{2^2 + 2^2 + (-3)^2}\sqrt{3^2 + (-3)^2 + 5^2}}$$

$$= \frac{15}{\sqrt{17}\sqrt{43}} = \frac{15}{\sqrt{731}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$$

Q5 Sol:Let θ be the angle between the line and the norma lto the plane.

Converting the given equations into vector form, wehave

$$\vec{r} = \left(-\hat{\imath} + 3\hat{k}\right) + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

$$and\ \vec{r}.\left(10\hat{\imath} + 2\hat{\jmath} - 11\hat{k}\right) = 3$$

$Here, \vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ and $\vec{n} = 10\hat{\imath} + 2\hat{\jmath} - 11\hat{k}$

$$sin\,\phi = \left|\frac{\left(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}\right).(10\hat{\imath} + 2\hat{\jmath} - 11\hat{k})}{\sqrt{2^2 + 3^2 + 6^2}\sqrt{10^2 + 2^2 + (-11)^2}}\right|$$

$$sin\,\phi = \left|\frac{-40}{7x5}\right| = \frac{8}{21} \Rightarrow \emptyset = sin^{-1}\left(\frac{8}{21}\right)$$

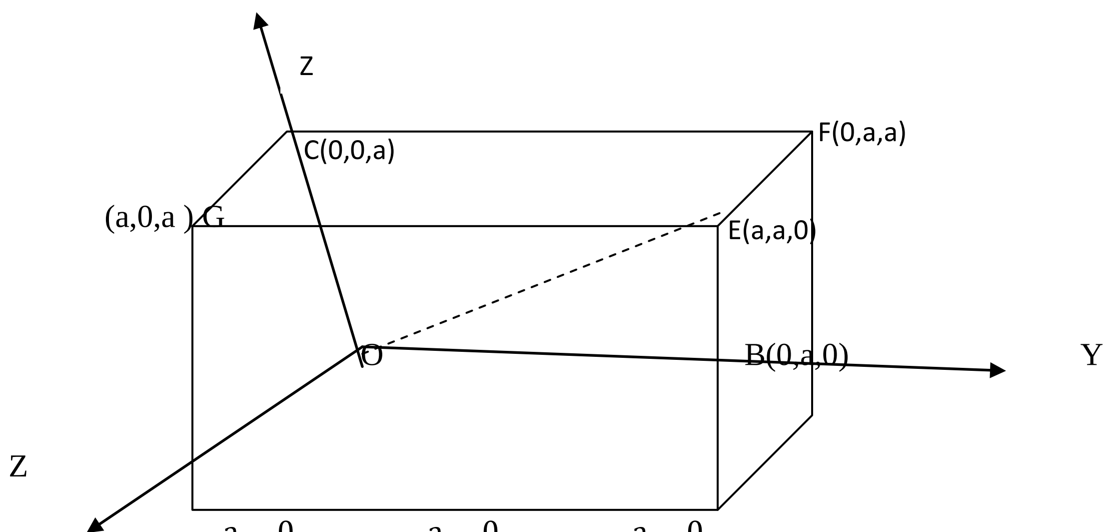
LEVEL-II

Q1 Sol: $-3 + 15 + 4p = 0$

$$4p = -12 \Rightarrow p = -3$$

Q2 Sol: Let OADBFEGC be the cube with each side a.

The four digonals OE, AF, BG & CD



$$\text{dC's of OE} = \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

$$= \frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}$$

Similarly dC's of AF, BG & CD are

$$= \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

Let l, m, n be the dC's of the given line which make sangle α, β, γ & δ $with$ OE, AF, BG & CD.

Then $cos\alpha = \frac{1}{\sqrt{3}}(l+m+n) \ldots\ldots (1), cos\beta = \frac{1}{\sqrt{3}}(-l+m+n) \ldots\ldots (2)$

$cos\gamma = \frac{1}{\sqrt{3}}(l-m+n) \ldots\ldots\ldots (3), cos\delta = \frac{1}{\sqrt{3}}(l+m-n) \ldots\ldots (4)$

Squaring & $add\ eq$ 1,2,3,4

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)] = \frac{4}{3} \qquad \because l^2+m^2+n^2 = 1$$

(vii) Distance of a point from a plane

LEVEL I

Q1 Sol:Distance of the plane $2x - y + 2z + 1 = 0$ from the origin is

$$d = \frac{1}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{1}{3}\text{unit}$$

Q2 Sol: Given line is $\frac{x-0}{\frac{1}{2}} = \frac{y-0}{\frac{1}{3}} = \frac{z-0}{\frac{1}{4}}$

$\therefore$ point is (0,0,0)

Q3 Sol: Here, $= 2\hat{i} + 5\hat{j} - 3\hat{k}$, $N = 6\hat{i} - 3\hat{j} + 2\hat{k}$ and $d = 4$.
Therefore, the distance of the point $(2, 5, -3)$ from the given plane is

$$\frac{(2\hat{i} + 5\hat{j} - 3\hat{k}).(6\hat{i} - 3\hat{j} + 2\hat{k}) - 4}{\sqrt{36^2 + 9^2 + 4^2}} = \frac{13}{7}$$

Q4Sol: Distance of the plane $2x - y + 2z + 1 = 0$ from the origin is

$$d = \frac{1}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{1}{3}\text{unit}$$

Q5 Sol: Distance $= \sqrt{(a-a)^2 + (b-0)^2 + (c-0)^2} = \sqrt{b^2 + c^2}$

LEVEL II

Q1 Sol: we have, $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$..................(1)

$\therefore$ any point on (1) is Q(-2+3λ,-1+2λ,3+2λ)

Point P(1,3,3)

$\therefore PQ = 5$

$\Rightarrow PQ^2 = 25$

$\Rightarrow (3 - 3\lambda)^2 + (4 - 2\lambda)^2 + (-2\lambda)^2 = 25$

$\Rightarrow 17\lambda^2 - 34\lambda = 0 \quad \Rightarrow \lambda = 0,2$

Required points are $(-2, -1, 3)$ & $(4, 3, 7)$

Q2 Sol:Eq of line AB is $\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{2}$

Eq of line PM ∥ to given line is

$$\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = \lambda \dots\dots\dots..(1)$$

Any pt M(say) on the line is

$x = \lambda + 3$; $y = 2\lambda + 4$; $z = 2\lambda + 5$

$\because$ this pt M is also on the given plane

$\therefore \lambda + 3 + 2\lambda + 4 + 2\lambda + 5 = 2$

$5\lambda = 2 - 12 = -10 \Rightarrow \lambda = -2$

$\therefore pt. M(-2+3, -4+4, -4+5) i.e M(1,0,1)$

$\therefore PM = \sqrt{(3-1)^2 + (4-0)^2 + (5-1)^2}$

$= \sqrt{4+16+16} = \sqrt{36}$

$\therefore PM = 6\, units$

Q3 Sol:Let A, B, C be dR's of Normal to the plane passing

through the pts $(3,-1,2), (5,2,4) \& (-1,-1,6)$

$\therefore A(x-3) + B(y+1) + C(z-2) = 0 \ldots \ldots \ldots \ldots \ldots (1)$

$\therefore A(5-3) + B(2+1) + C(4-2) = 0$

$= 2A + 3B + 2C = 0 \ldots \ldots .. (2)$

$\& = -4A + 0B + 4C = 0 \ldots \ldots \ldots \ldots \ldots \ldots (3)$

$$\therefore \frac{A}{12-0} = \frac{-B}{8+8} = \frac{C}{0+12} \Longrightarrow \frac{A}{12} = \frac{B}{-16} = \frac{C}{12}$$

$$\Longrightarrow \frac{A}{3} = \frac{B}{-4} = \frac{C}{3}$$

$\therefore plane\ is$

$\Longrightarrow 3(x-3) - 4(y+1) + 3(z-2) = 0$

$\Longrightarrow 3x - 4y + 3z - 9 - 4 - 6 = 0$

$\therefore 3x - 4y + 3z = 19$

$\therefore$ distance (d) of that pt(6,5,9) from this plane is

$$d = \left|\frac{3X6 - 4X5 + 3X9 - 19}{\sqrt{9+16+9}}\right| = \left|\frac{18 - 20 + 27 - 19}{\sqrt{34}}\right| = \frac{6}{\sqrt{34}} units$$

Q4 Sol: we have, line

$\vec{r} = 2\,\hat{i} - \hat{j} + 2\,\hat{k} + \lambda(3\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}) \ \ldots \ldots .. (1)$

and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \ldots \ldots .. (2)$

$\therefore point\ of\ intersection\ of (1) \& (2)\ is$

$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - \hat{j} + \hat{k}) = 5$

$$\Rightarrow (2 + 1 + 2) + \lambda(3 - 4 + 2) = 0$$

$$\Rightarrow 5 + \lambda = 5 \Rightarrow \lambda = 0$$

$(1) \Rightarrow$ point of intersection is$(2, -1,2)$

Required distance between (-1,-5,-10) and $(2, -1,2)$ is

$$\sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$$

LEVEL III

Q1 Sol:Eq of line PM is

$$\frac{x - 1}{2} = \frac{y - 3}{-1} = \frac{z - 4}{1} = \lambda(\text{say})$$

$\therefore$ co − ordinates of M are

$x = 2\lambda + 1, y = -\lambda + 3, z = \lambda + 4$

$\because$ pt $M(2\lambda + 1, -\lambda + 3, \lambda + 4)$is on the plane $2x - y + z + 3 = 0$

$2(2\lambda + 1) - (-\lambda + 3) + (\lambda + 4) + 3 = 0$

$\Rightarrow 4\lambda + \lambda + \lambda = -2 + 3 - 4 - 3 = -6$

$\therefore \lambda = -1$

$\therefore$ pt $M(-1,4,3)$ is the foot of the $\perp$ from the pt$(1,3,4)$on a plane.

Let $P'(\alpha, \beta, \gamma)$is the image of $P(1,3,4)$ in the plane

$$\therefore \frac{1 + \alpha}{2} = -1, \frac{3 + \beta}{2} = 4, \frac{4 + \gamma}{2} = 3$$

$\alpha = -2 - 1 = -3, \beta = 8 - 3 = 5, \gamma = 6 - 4 = 2$

$\therefore$ image is$(-3,5,2)$

$\therefore PM = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}$ units

Q2 Sol:Let A, B, C be dR′s of Normal to the plane passing

through the pts $(3, -1,2), (5,2,4) \& (-1, -1,6)$

$\therefore A(x - 3) + B(y + 1) + C(z - 2) = 0 \ldots \ldots \ldots \ldots \ldots (1)$

$\therefore A(5-3)+B(2+1)+C(4-2)=0$

$= 2A+3B+2C=0 \ldots\ldots..(2)$

$\& = -4A+0B+4C=0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$

$\therefore \frac{A}{12-0}=\frac{-B}{8+8}=\frac{C}{0+12} \Longrightarrow \frac{A}{12}=\frac{B}{-16}=\frac{C}{12}$

$\Longrightarrow \frac{A}{3}=\frac{B}{-4}=\frac{C}{3}$

$\therefore$ plane is

$\Longrightarrow 3(x-3)-4(y+1)+3(z-2)=0$

$\Longrightarrow 3x-4y+3z-9-4-6=0$

$\therefore 3x-4y+3z=19$

$\therefore$ distance (d)of that pt(6,5,9)from this planeis

$$d=\left|\frac{3X6-4X5+3X9-19}{\sqrt{9+16+9}}\right|=\left|\frac{18-20+27-19}{\sqrt{34}}\right|=\frac{6}{\sqrt{34}}\text{units}$$

Q3 Sol:The eq of the plane containing the lines

$\vec{r}=(\hat{\imath}+\hat{\jmath}+\hat{k})+\lambda(\hat{\imath}+2\hat{\jmath}-\hat{k})$

$\&\vec{r}=(\hat{\imath}+\hat{\jmath}+0\hat{k})+\mu(-\hat{\imath}+\hat{\jmath}-2\hat{k})$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-0 \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix}=0$$

$$\therefore (x-1)(-4+1)+(y-1)(-2-1)+(z-0)(1+2)=0$$

$$\Rightarrow -3x+3-3y+3+3z-0=0 \Rightarrow 3x+3y-3z=6$$

$\Rightarrow x+y-z=2 \ldots\ldots\ldots\ldots\ldots.(1)$

$\therefore$ Distance of the plane (1)fromthe origin is

$d=\left|\frac{-2}{\sqrt{1+1+1}}\right|=\frac{2}{\sqrt{3}}\text{units}$

Distance of plane 1 from(1,1,1)is

$d = \left|\frac{1+1-1-2}{\sqrt{1+1+1}}\right| = \frac{1}{\sqrt{3}}\text{units}$

(viii) Equation of a plane through the intersection of two planes

LEVEL III

Q1 Sol:Let A, B, C are the dR′s of Normal to the plane passing through pt. (1,2,1)

$\therefore A(x-1) + B(y-2) + C(z-1) = 0 \ldots\ldots..(i)$

$\therefore$ the plane(i) is $\perp$ to the line joining the pts(1,4,2) & (2,3,5)

$\therefore$ dR′s of line 1, −1,3

$\therefore \frac{A}{1} = \frac{B}{-1} = \frac{C}{3} = k$

$\therefore k(x-1) - k(y-2) + 3k(z-1) = 0$

$\therefore x - y + 3z - 1 + 2 - 3 = 0$

$\Longrightarrow x - y + 3z = 2$ is req. plain

Let d is distance of this plane from origin

$\therefore d = \left|\frac{-2}{\sqrt{1+1+9}}\right| = \frac{2}{\sqrt{11}}\text{units}$

Q2 Sol:The plane conating the line of intersection of the planes

$x + 2y + 3z - 4 = 0$ & $2x + y - z + 5 = 0$ *is*

$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$

$$\Longrightarrow (1+2\lambda)x + (2+\lambda)y + (3-\lambda)z - 4 + 5\lambda = 0 \ldots\ldots\ldots\ldots (i)$$

Now, sincetheplane(i)is $\perp$ totheplane

$$\Longrightarrow 5x + 3y + 6z + 8 = 0$$

$\therefore 5(1+2\lambda) + 3(2+\lambda) + 6(3-\lambda) = 0$

$\therefore 10\lambda + 3\lambda - 6\lambda = -5 - 6 - 18 \Longrightarrow 7\lambda = -29 \Longrightarrow \lambda = \frac{-29}{7}$

$\therefore \left(1 - \frac{58}{7}\right)x + \left(2 - \frac{29}{7}\right)y + \left(3 + \frac{29}{7}\right)z - 4 + 5\left(\frac{-29}{7}\right) = 0$…..(ii)

$$\Longrightarrow -51x - 15y + 50z - 28 - 145 = 0$$

$\therefore 51x + 15y - 50z + 173 = 0$

Q3 Sol:Let a, b, c be the dR′s of Normal to the plane

$\therefore$ eq. of the plane containing the pt$(1, -1, 2)$ is

$a(x-1) + b(y+1) + c(z-2) = 0 \ldots\ldots\ldots (1)$

$\because$ plane(1)is $\perp$ to both given planes

$$\therefore 2a + 3b - 2c = 0$$

$\Rightarrow a + 2b - 3c = 0$

$$\therefore \frac{a}{-9+4} = \frac{-b}{-6+2} = \frac{c}{4-3} \qquad \Rightarrow \frac{a}{-5} = \frac{b}{4} = \frac{c}{1}$$

$\therefore$ plane is

$= \quad -5(x-1) + 4(y+1) + 1(z-2) = 0$

$= -5x + 5 + 4y + 4 + z - 2 = 0 \Longrightarrow 5x - 4y - z = 7$

LEVEL III

Q1 Sol: Let A, B, C be the dR′s of Normal to the required plane

$\therefore$ the plane passing through the pt$(1,1,1)$ is

$A(x-1) + B(y-1) + C(z-1) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (i)$

Since the plane(i) contains line

$$\vec{r} = \left(-3\hat{i} + \hat{j} + 5\hat{k}\right) + \lambda(3\hat{i} - \hat{j} - 5\hat{k})$$

$\therefore A(-3-1) + B(1-1) + C(5-1) = 0$

$\Longrightarrow -4A + 0B + 4C = 0 \qquad \ldots\ldots\ldots\ldots (ii)$

$\& \ 3A - B - 5C = 0 \ldots\ldots\ldots\ldots\ldots (iii)$

$$\therefore \frac{A}{0+4} = \frac{-B}{20-12} = \frac{C}{4-0} \Longrightarrow \frac{A}{4} = \frac{B}{-8} = \frac{C}{4}$$

$$\therefore \frac{A}{1} = \frac{B}{-2} = \frac{C}{1}$$

$\therefore$ Equation of plane is $1(x-1) - 2(y-1) + 1(z-1) = 0$

$= x - 2y + z - 1 + 2 - 1 = 0 \Rightarrow x - 2y + z = 0$

$\therefore$ Equation of plane is $x - 2y + z = 0$

Q2Sol: equation of palne through$(1,1,1\ )$is

$a(x-1) + b(y-1) + c(z-1) = 0$……………..(1)

Applying the condition of perpendicularity to the planegiven in(1)

with the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.
We have, a+2b+3c=0………(2)

2a-3b+4c=0……….(3)

Solving (2) & (3) , we get

$$\frac{a}{17} = \frac{b}{2} = \frac{c}{-7}$$

$(1) \Rightarrow 17(x-1) + 2(y-1) - 7(z-1) = 0$

$\Rightarrow 17x + 2y - 7z = 12$

Q3 Sol:Eq of plain conatining the pts $A(0,0,2)$& $B(3,-1,2)$& $\parallel$ *to* the line is

$$\begin{vmatrix} x & y & z \\ 3 & -1 & 2 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow x(-7+8) - y(21-2) + z(-12+1) = 0$$

$$\Rightarrow x - 19y - 11z = 0$$

Q4 Sol: Any pt. Q(say) on the given line is $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$

$\therefore$ Co − ordinates of Q are $x = \lambda - 5, y = 4\lambda - 3, z = -9\lambda + 6$

Now dR′s of line $\perp$ PQ are $\lambda - 5 - 2, 4\lambda - 3 - 4, -9\lambda + 6 + 1$

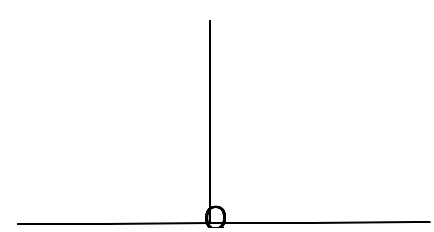
$$<, \lambda - 7, 4\lambda - 7, -9\lambda + 7 >$$

$\because$ PQ $\perp$ to the given line P(2,4,-1)

$\therefore 1(\lambda - 7) + 4(4\lambda - 7) - 9(-9\lambda + 7) = 0$

$\Rightarrow \lambda + 16\lambda + 81\lambda = 7 + 28 + 63 = 98$

$\Rightarrow 98\lambda = 98 \Rightarrow \lambda = 1$



$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$$

$\therefore$ pt. $Q(-4,1,-3)$

$\therefore$ Eqof $\perp$ PQis $\frac{x-2}{-6}=\frac{y-4}{-3}=\frac{z+1}{-2}$

$\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$

(ix)Foot of perpendicular and image with respect to a line and plane

LEVEL II

Q1 Sol:Eq of line is

$$\therefore \frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}=\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=\lambda \ldots\ldots\ldots.(1)$$

Eq of the plane conatining the pts $(1,2,3),(2,2,1)\&(-1,3,6)$

$$=\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 2-2 & 1-3 \\ -1-1 & 3-2 & 6-3 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 0 & -2 \\ -2 & 1 & 3 \end{vmatrix}=0 \Rightarrow \therefore -2x+y+z-3=0 \;\ldots.(2)$$

Any pt on the line(1) is

$x=-\lambda+3\;;y=\lambda-4\;;z=6\lambda-5$

$\because$ this pt is also on the plane(2)

$-2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-3=0$

$\therefore 2\lambda+\lambda+6\lambda=6+4+8=18$

$\therefore \lambda=2$

$\therefore$ the pt of intersection of line &*plane* is $(1,-2,7)$

$x=-2+3=1$

$y=2-4=-2$

$z=12-5=7$

Ans: (1,-2, 7)

Q2 Sol: Any pt the M(say) be foot of $\perp$ from P on the given line $\therefore$ M

$x = 3\lambda + 6$; $y = 2\lambda + 7$; $z = -2\lambda + 7$

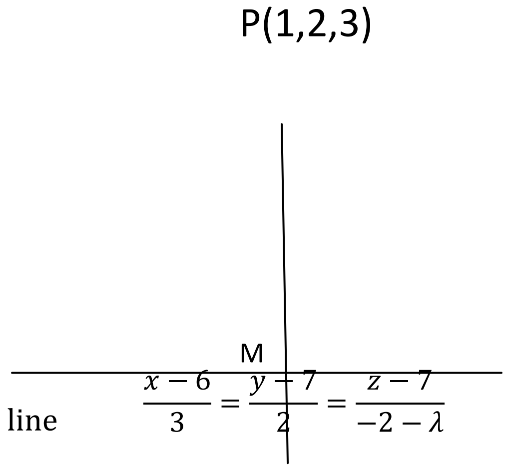
DR′s of PM are $(3\lambda + 5, 2\lambda + 5, -2\lambda + 4)$

$\therefore 3(3\lambda + 5) + 2(2\lambda + 5) - 2(-2\lambda + 4) = 0$

$= 9\lambda + 4\lambda + 4\lambda = -15 - 10 + 8 = -17$

$\Rightarrow 17\lambda = -17 \Rightarrow \lambda = -1$

$\therefore$ the pt M(3,5,9)



Let A, B, C be the DR′s of Normal to the plane conatining the line

& *the pt*(1,2,3) is

$A(x - 1) + B(y - 2) + C(z - 3) = 0$ … … … … … … … (1)

$\because$ the plane(1) is conating the given line

$\therefore A(6 - 1) + B(7 - 2) + C(7 - 3) = 0$

$5A + 5B + 4C = 0$ … … … … … . . (2)

Also $3A + 2B - 2C = 0$ … … … . . (3)

$$\therefore \frac{A}{-10-8} = \frac{-B}{-10-12} = \frac{C}{10-15} \Rightarrow \frac{A}{-18} = \frac{B}{22} = \frac{C}{-5}$$

$\therefore$ planeis $-18(x - 1) + 22(y - 2) - 5(z - 3) = 0$

$\Rightarrow 18x - 22y + 5z + 11 = 0$

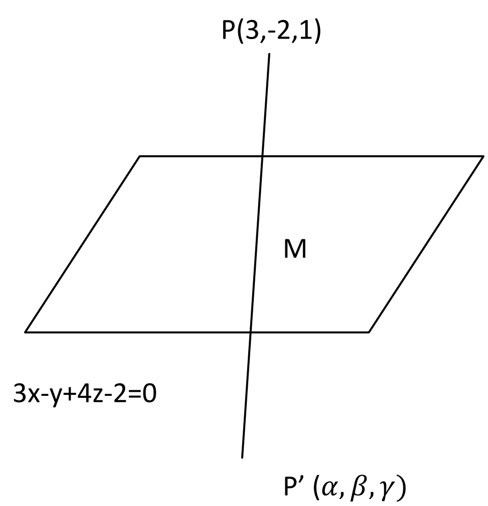
Q3 Sol:Eq of line PM is $\frac{x-3}{3} = \frac{y+2}{-1} = \frac{z-1}{4} = \lambda$

Any pt. M(say) on PP′is

$x = 3\lambda + 3$; $y = -\lambda - 2$; $z = 4\lambda + 1$

$\because$ the pt M is also on the given plane

$\therefore 3(3\lambda + 3) - (-\lambda - 2) + 4(4\lambda + 1) - 2 = 0$



$\Rightarrow 9\lambda + \lambda + 16\lambda = -9 - 2 - 4 + 2 = -13$

$\Rightarrow 26\lambda = -13 \Rightarrow \lambda = -\frac{1}{2}$

$\therefore$ pt M $\left(\frac{-3}{2}+3, \frac{1}{2}-2, -2+1\right)$

$\therefore$ pt M $= \left(\frac{3}{2}, \frac{-3}{2}, -1\right)$

If $P'(\alpha, \beta, \gamma)$is the image of the point $P(3, -2, 1)$

$\therefore \frac{3+\alpha}{2} = \frac{3}{2}, \frac{\beta-2}{2} = \frac{-3}{2}, \frac{1+\gamma}{2} = -1$

$\therefore \alpha = 0, \beta = -1, \gamma = -3$

$\therefore$ ptis$(0, -1, -3)$

LEVEL-III

Q1 Sol: Equation of line BC $= \frac{x-4}{1} = \frac{y-7}{2} = \frac{z-1}{-2} = \lambda$

Coordinate of M are $(4+\lambda, 7+2\lambda, 1-2\lambda)$

Dr's of M $< 3+\lambda, 7+2\lambda, -2-2\lambda >$

Dr's of BC <1,2,-2>

$\therefore AM \perp BC \implies (3+\lambda) + 2(7+2\lambda) - 2(-2-2\lambda) = 0$

$$\implies \lambda = \frac{-7}{3}$$

Coordinate of foot M$\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$

Q2 Sol: Let $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z+3}{2} = \lambda$

$\therefore$ co − ordinatesofMare$(2+3\lambda, -1-\lambda, -3+2\lambda)$

Dr's of PM is $< 1+3\lambda, 1-\lambda, -4+2\lambda >$

Dr's of line <3,-1,2>

$\therefore$ line is $\perp$ toPM

$\therefore 3(1+3\lambda) - (1-\lambda) + 2(-4+2\lambda) = 0$

$\implies \lambda = \frac{3}{7}$

$\therefore$ co − ordinate of M are $(2+3.\frac{3}{7}, -1-\frac{3}{7}, -3+2.\frac{3}{7})$

i.e M $\left(\frac{23}{7}, \frac{-10}{7}, \frac{-15}{7}\right)$

Let $P(\alpha, \beta, \gamma)$ be the image of point P on line

$\therefore$ M is the mid point of PP′

$\therefore \dfrac{1+\alpha}{2}=\dfrac{23}{7}$; $\dfrac{-2+\beta}{2}=\dfrac{-10}{7}$; $\dfrac{1+\gamma}{2}=\dfrac{-15}{7}$

$\alpha=\frac{39}{7}$; $\beta=\frac{-6}{7}$; $\gamma=\frac{-37}{7}$

Required image is $\left(\frac{39}{7}\ ,\frac{-6}{7},\frac{-37}{7}\right)$

Q3 Sol: Vector normal to plane $\vec{n}=12\hat{i}-4\hat{j}+3\hat{k}$

Equation of plane through (12,-4,3) is

$(\vec{r}-\vec{a}).\vec{n}=0$

$\therefore\ \vec{a}=\ 12\hat{i}-4\hat{j}+3\hat{k}$

$\therefore [\vec{r}-(12\hat{i}-4\hat{j}+3\hat{k})].(12\hat{i}-4\hat{j}+3\hat{k})=0$

$\vec{r}.(12\hat{i}-4\hat{j}+3\hat{k})=169$

put $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then,

$12x-4y+3z=169$

Q4 Sol:Eq of line PM is

$$\frac{x-3}{2}=\frac{y-2}{-1}=\frac{z-1}{1}=\lambda(\text{say})$$

$\therefore$ co − ordinates of M are

$x=2\lambda+3, y=-\lambda+2, z=\lambda+1$

$\because$ pt $M(2\lambda+3,-\lambda+2,\lambda+1)$ is on the plane $2x-y+z+1=0$

$2(2\lambda+3)—(-\lambda+2)+(\lambda+1)+1=0$

$\Rightarrow 4\lambda+\lambda+\lambda+6-2+1+1=0$

$\therefore \lambda=-1$

$\therefore$ pt M(1,3,0) is the foot of the $\perp$ from the pt(3,2,1)on a plane.

Let $P'(\alpha,\beta,\gamma)$ is the image of P(1,3,4) in the plane

$$\therefore \frac{3+\alpha}{2} = 1, \frac{2+\beta}{2} = 3, \frac{1+\gamma}{2} = 0$$

$\alpha = 2 - 3 = -1, \beta = 6 - 2 = 4, \gamma = -1$

$\therefore$ image is $(-1,4,-1)$

$\therefore PM = \sqrt{(4)^2 + (-2)^2 + 2^2} = \sqrt{24} = 2\sqrt{6} units$

CHAPTER-11

LINEAR PROGRAMMING

(i) LPP and its Mathematical Formulation

LEVEL I

Q1 Sol: Let the dietician mix x Kg of food 'I' with y kg of food 'II'. Then, the mathematical model of the LPP is as follows:

$$\text{Minimize} Z = 50x + 70y$$

$$\text{Subject to } 2x + y \geq 8$$

$$x + 2y \geq 10$$

$$x, y \geq 0$$

(ii) Graphical method of solving LPP (bounded and unbounded solutions)

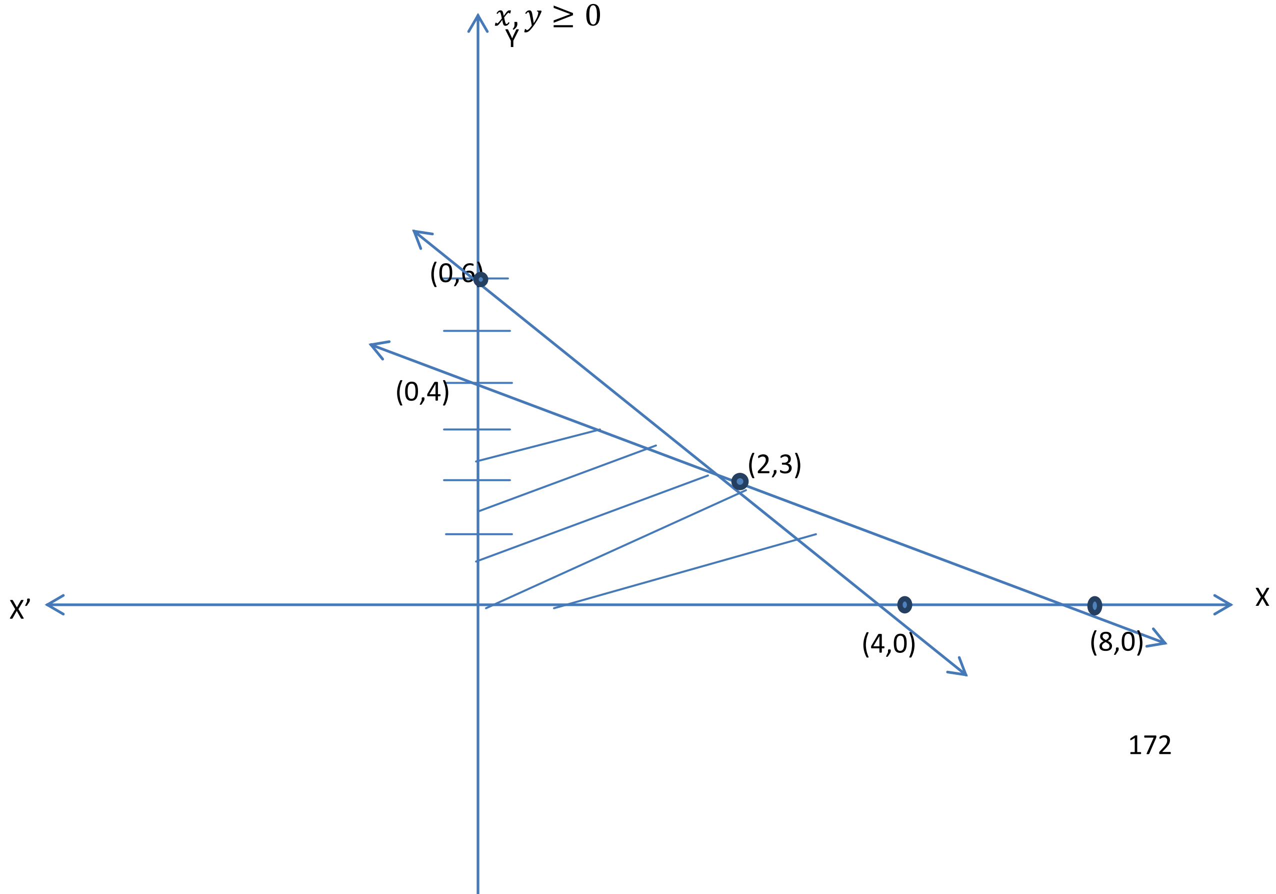
LEVEL I

Q1. Minimize $Z = -3x + 4y$

Subject to $x + 2y \leq 8$…… (i)

$3x + 2y \leq 12$ …….. (ii)

$x, y \geq 0$



Corner Pts.	Z
(0,0)	0
(0,4)	16
(4,0)	-12(Minimum)
(2,3)	6

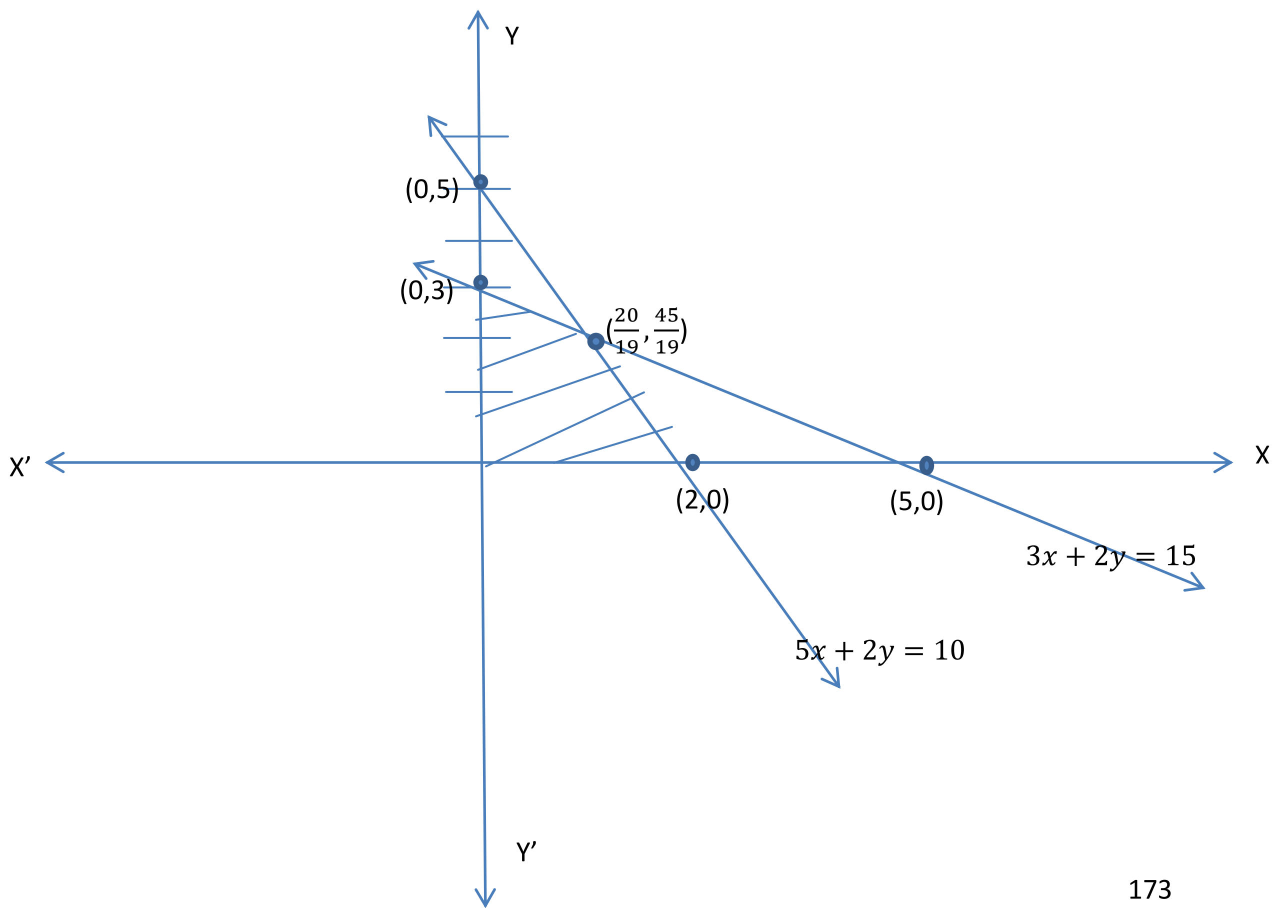
Y’

Q2.Sol Maximize $Z = 5x + 3y$

Subject to $3x + 5y \leq 15$…… (i)

$5x + 2y \leq 10$ …….. (ii)

$x \geq 0, y \geq 0$



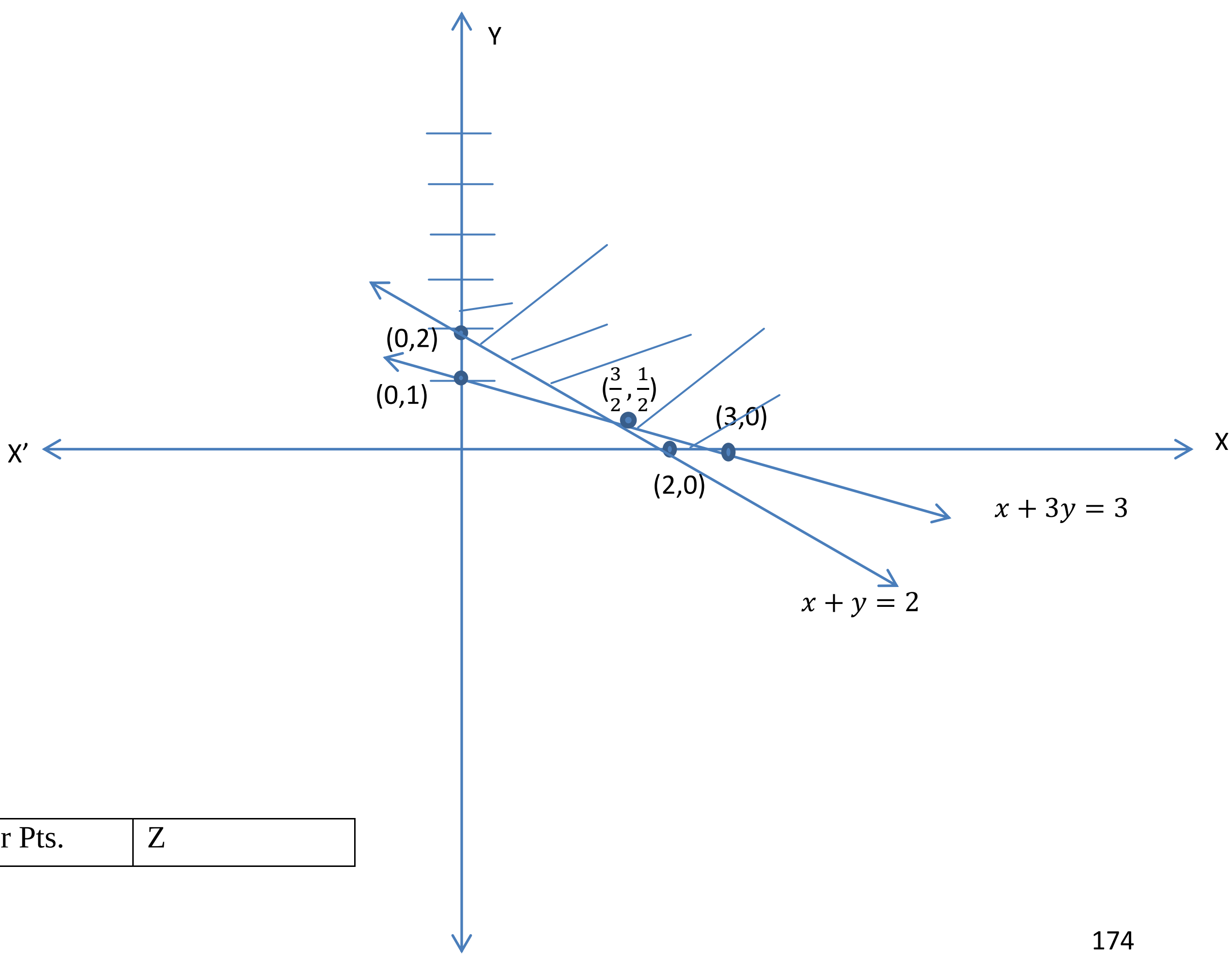
Corner Pts.	Z
(0,0)	0
(0,3)	9
(2,0)	10
$(\frac{20}{19}, \frac{45}{19})$	235/19(Maximum)

Q3. Sol: Minimize $Z = 3x + 5y$

Subject to $x + 3y \geq 3$…… (i)

$x + y \geq 2$ …….. (ii)

$x \geq 0, y \geq 0$



Corner Pts.	Z

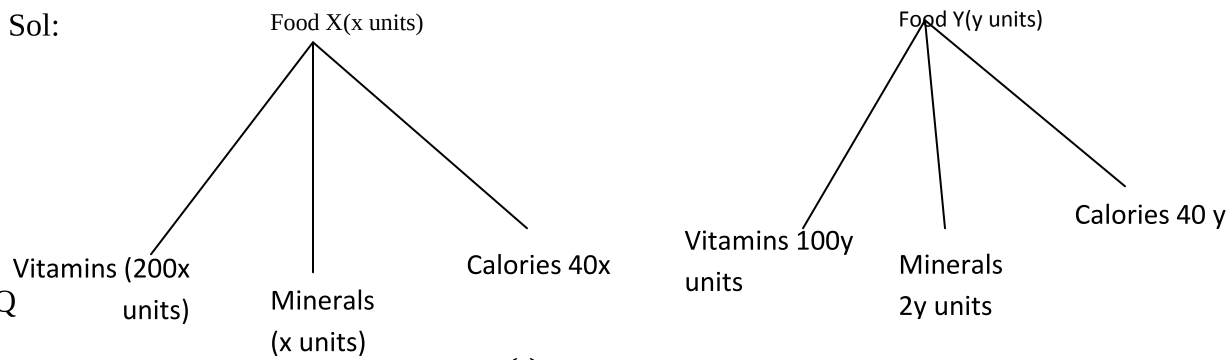
(0,2)	10
(3,0)	9
$(\frac{3}{2},\frac{1}{2})$	7(Minimum)

Y’

Diet problem

LEVEL II

Q1 Sol:



A/Q

$200x + 100y \geq 4000 \Rightarrow 2x + y \geq 40 \ldots (i)$

$x + 2y \geq 50 \ \ldots \ldots \ldots (ii)$

$$40x + 40y \geq 1400 \ \Rightarrow x + y \geq 35 \ \ldots (iii)$$

$\therefore$ The given L.P.P is Minimize $z = 4x + 3y$

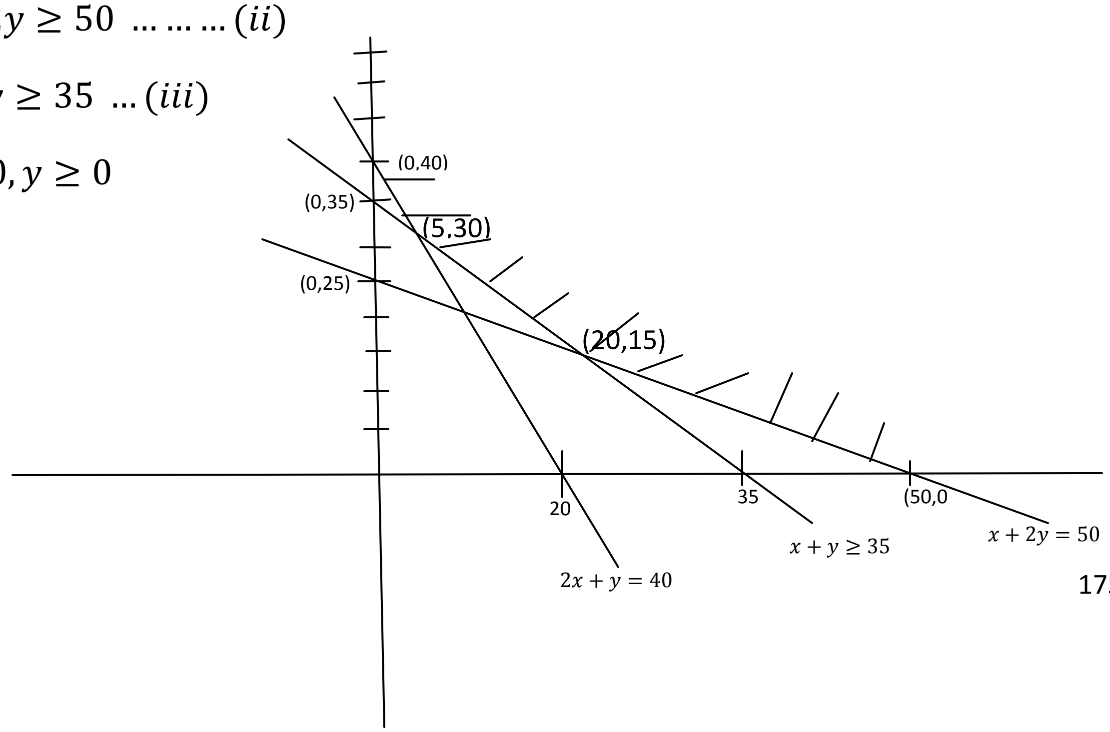
Subject to the constraints

$\Rightarrow 2x + y \geq 40 \ldots (i)$

$\Rightarrow x + 2y \geq 50 \ \ldots \ldots \ldots (ii)$

$\Rightarrow x + y \geq 35 \ \ldots (iii)$

$\Rightarrow x \geq 0, y \geq 0$



Corner pt. $z = 4x + 3y$

(0,40)	120
(5,30)	110 (minimum)
(20,15)	125
(50,0)	200

$\therefore z is min. at (5,30)$

$\therefore Food X = 5 units \& Food Y = 30\ units$

$$Minimum Cost = Rs\ 110$$

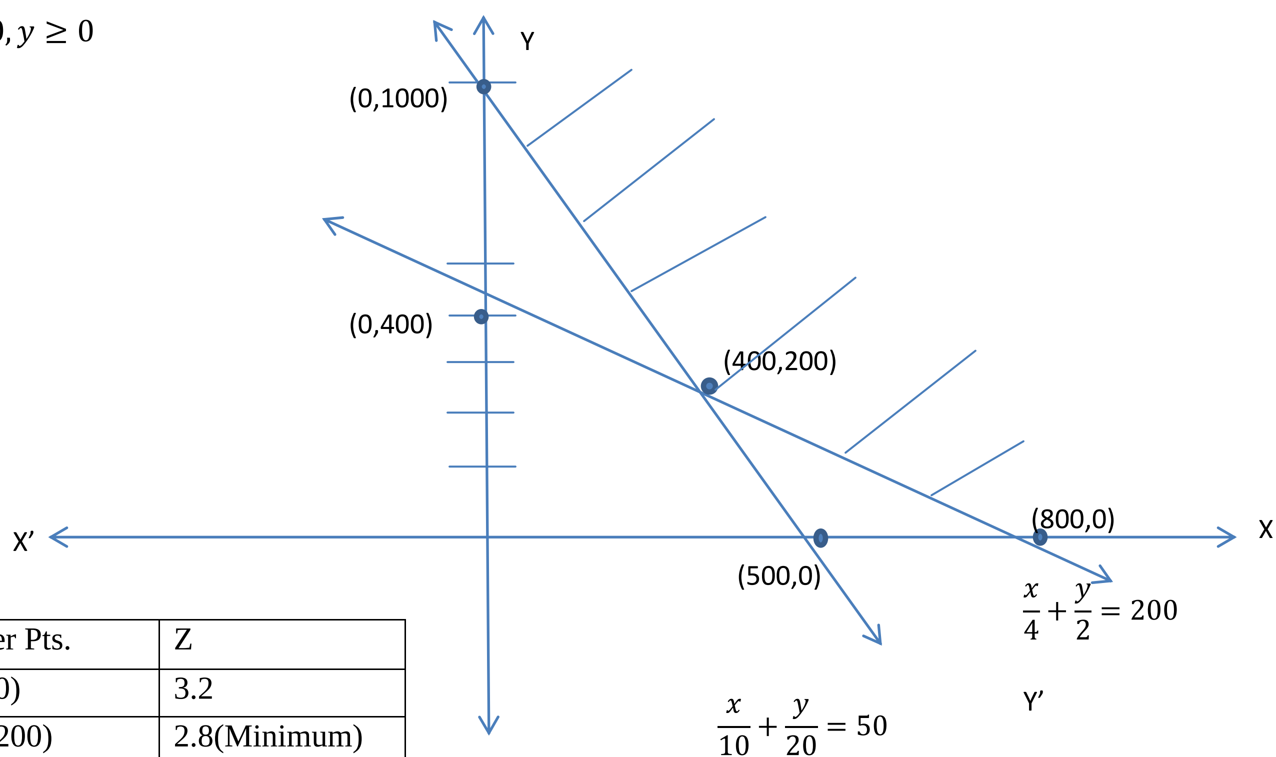
Q2. Sol: Suppose x grams of wheat &y grams of rice are mixed in the daily diet

The given L.P.P is minimize $z = \frac{4x}{1000} + \frac{6y}{1000}$

Subject to $\frac{x}{10} + \frac{y}{20} \geq 50 \ldots\ldots\ldots\ldots (i)$

$\frac{x}{4} + \frac{y}{2} \geq 200 \ldots\ldots\ldots\ldots. (ii)$

$x \geq 0, y \geq 0$



Corner Pts.	Z
(800,0)	3.2
(400,200)	2.8(Minimum)
(0,1000)	6

MANUFACTURING PROBLEM

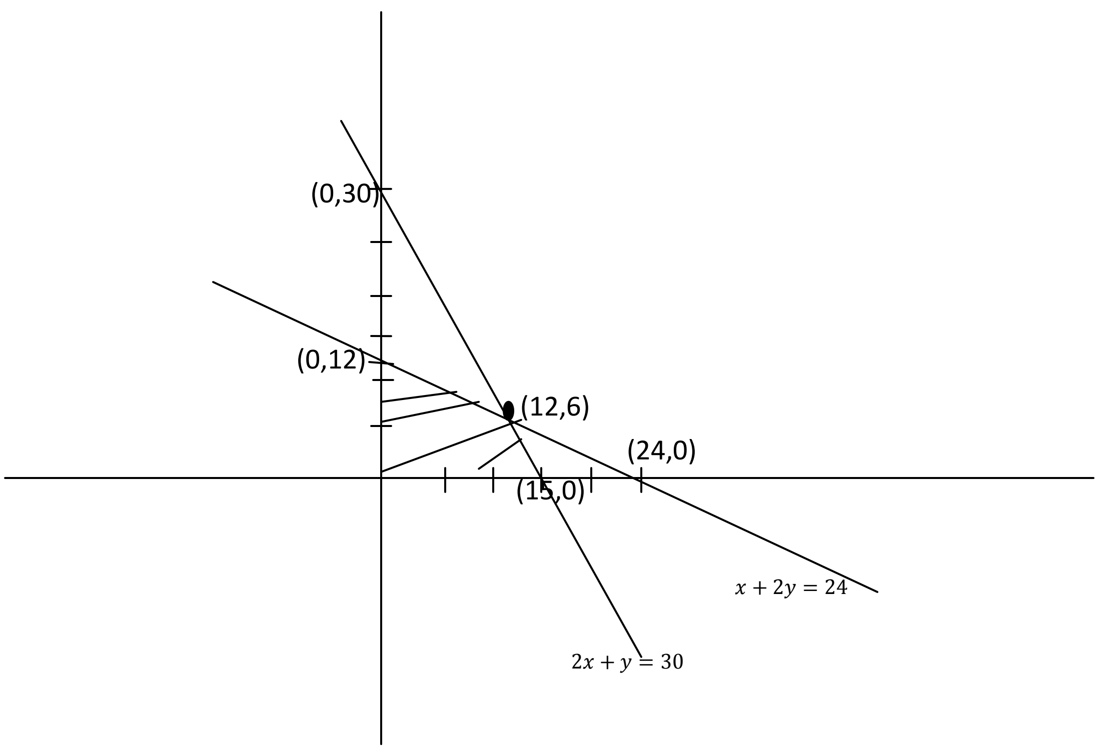
LEVEL II

Q1. Sol: The given L.P.P is maximize $z = 6x + 8y$

Subject to $2x + y \leq 30 \ldots\ldots\ldots\ldots (i)$

$$x + 2y \leq 24 \ldots\ldots\ldots\ldots. (ii)$$

$$x \geq 0, y \geq 0$$



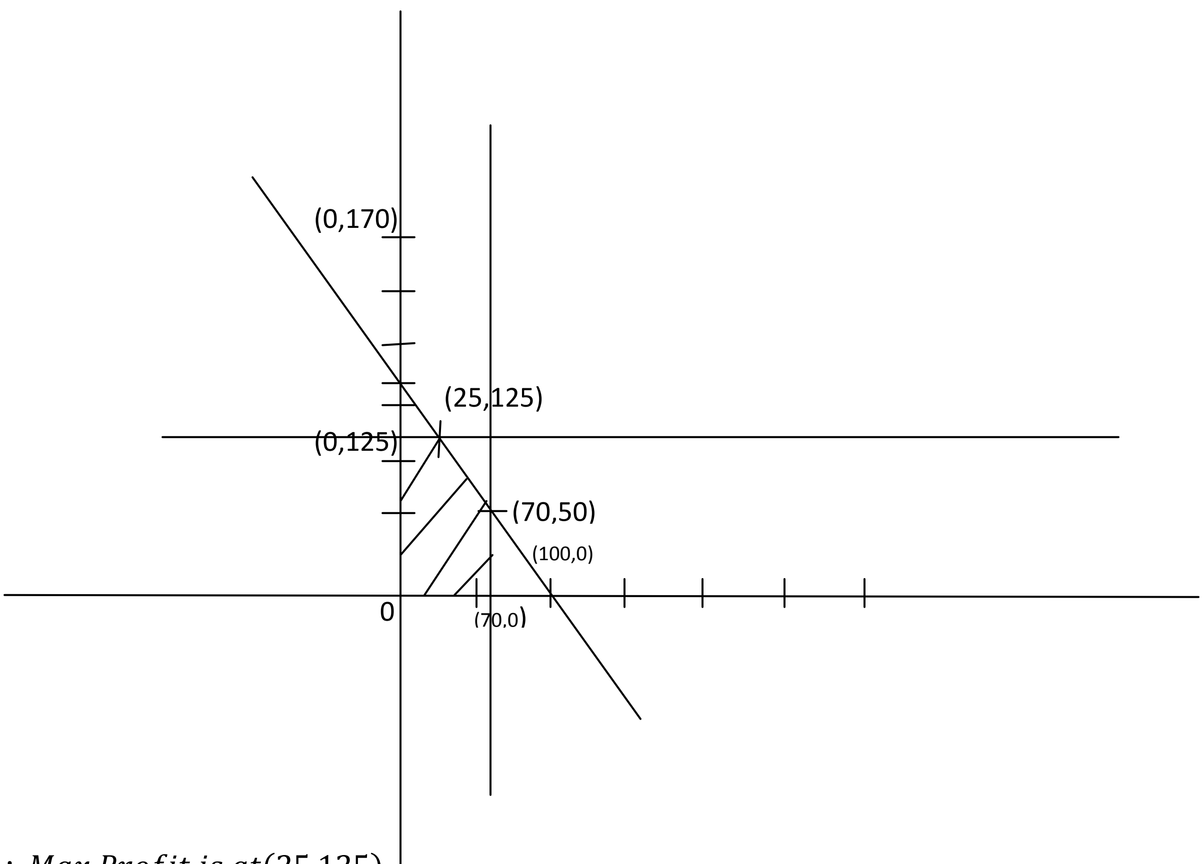
$\therefore$ Maximum Profit is Rs 120 for 12 units of article A & 6 units of article B

Q2The given L.P.P is maximize $z = 20x + 15y$

Subject to $5x + 3y \leq 500 \ldots\ldots\ldots\ldots (i)$

$$x \leq 70, y \leq 125$$

$$x \geq 0, y \geq 0$$



$\therefore Max\ Profit\ is\ at (25,125)$

$\therefore Product\ A = 25 units\ \&\ Product\ B = 125\ units$

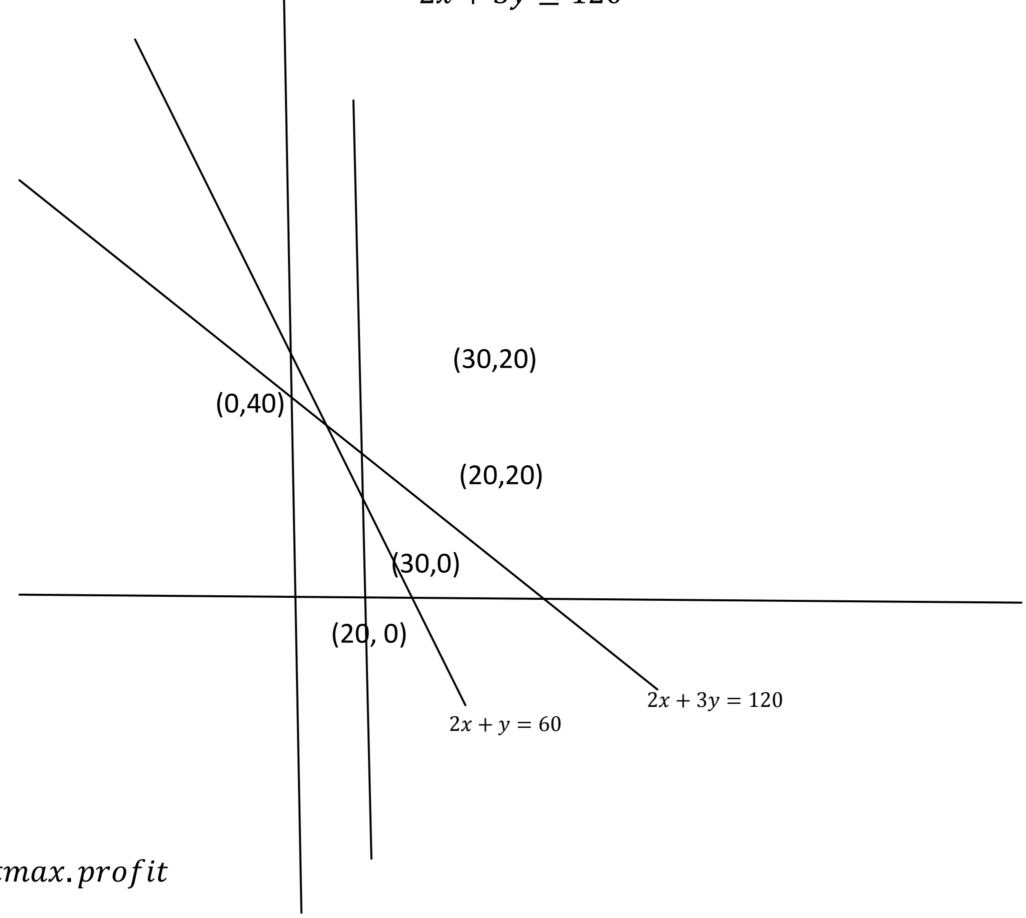
LEVEL III

Q1: Sol: The given L.P.P is maximize $z = \frac{75x}{100} + \frac{50y}{100}$

Subject to $2x + y \leq 60$

$$x \leq 20$$

$$2x + 3y \leq 120$$



$\therefore To set max. profit$

$$Cup A = 15\ units \& cup B = 30 units$$

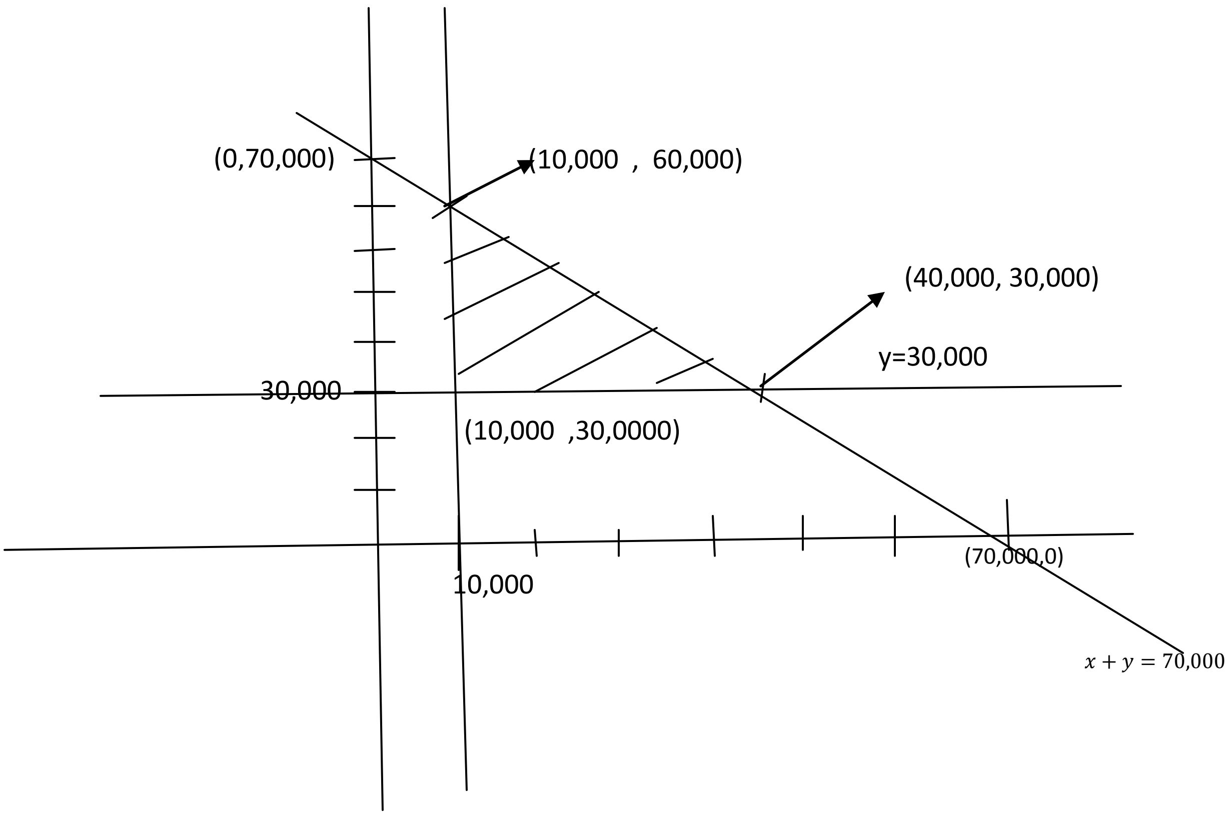
ALLOCATION PROBLEM

LEVEL II

Q1 Sol: The given L.P.P is max$z = \frac{8x}{100} + \frac{10y}{100}$

Subject to $x + y \leq 70{,}000$

$$x \geq 10{,}000, \quad y \geq 30{,}000$$



Max. Annual income =Rs 6800

Investment in Bond A=Rs 10,000

Investment in Bond B=Rs 60,000

Q2 Sol: The given L.P.P is $Min z = 400x + 300y$

Subject to $x + 2y \geq 120$

$$3x + 4y \geq 200$$

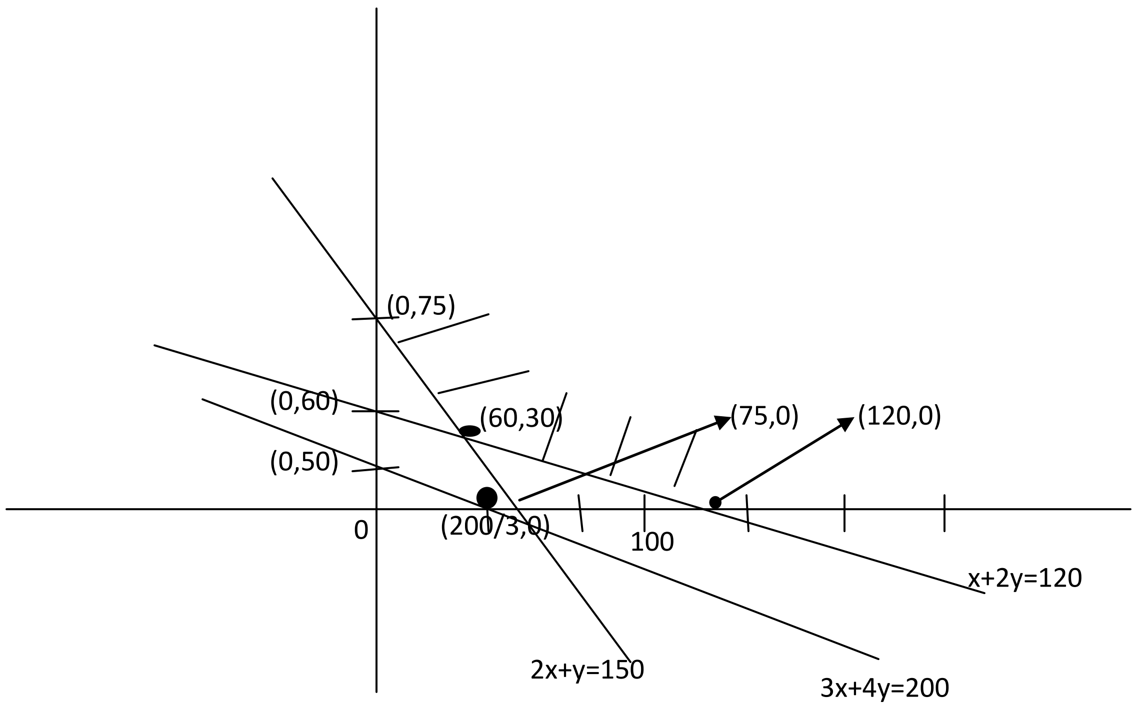
$$2x + y \geq 150$$

$$x \geq 0, y \geq 0$$

To minimize the cost

Refinery A runs =60days

Refinery B runs =30days



LEVEL III

Q1 Sol: The given L.P.P is maximize $z = 500x + 350y$

Subject to $x + y \leq 250$

$x \geq 25$

$y \geq 3x$

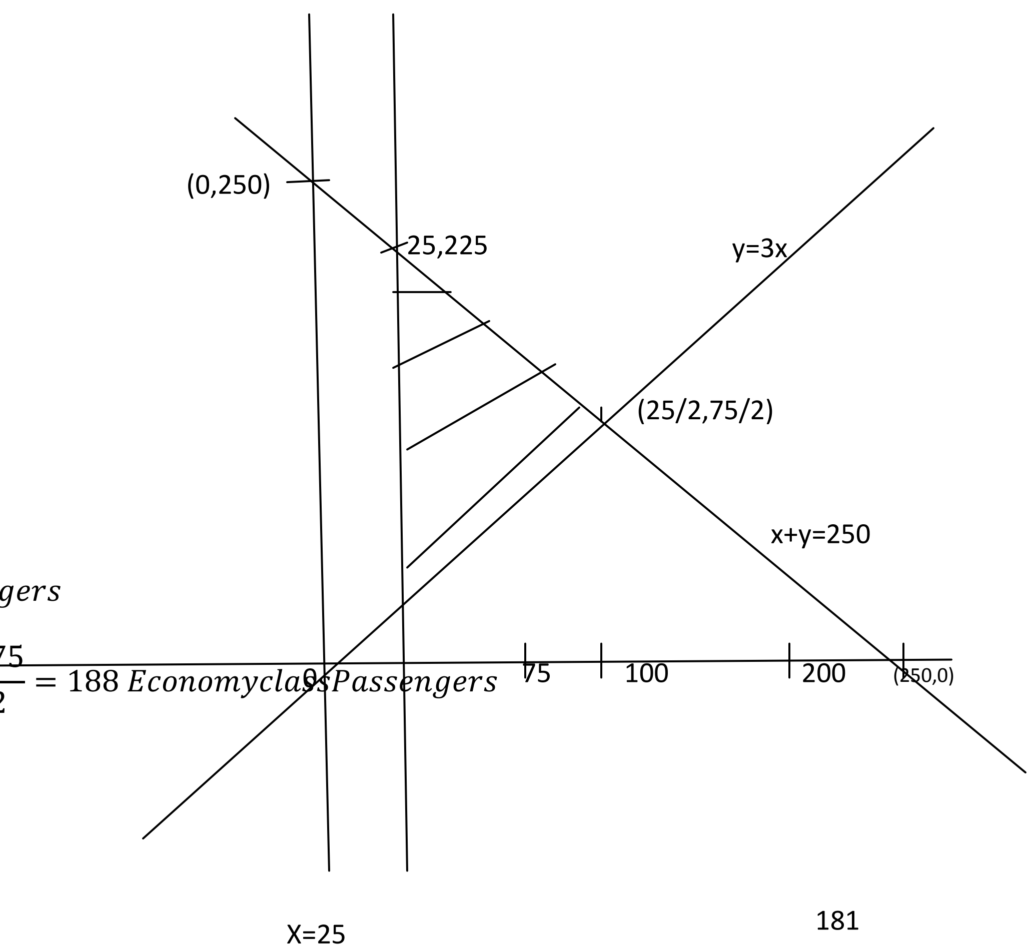
$x \geq 0, y \geq 0$

To set max Profit

There should be $\frac{125}{2}$

$= 62\ executiveclasspassengers$

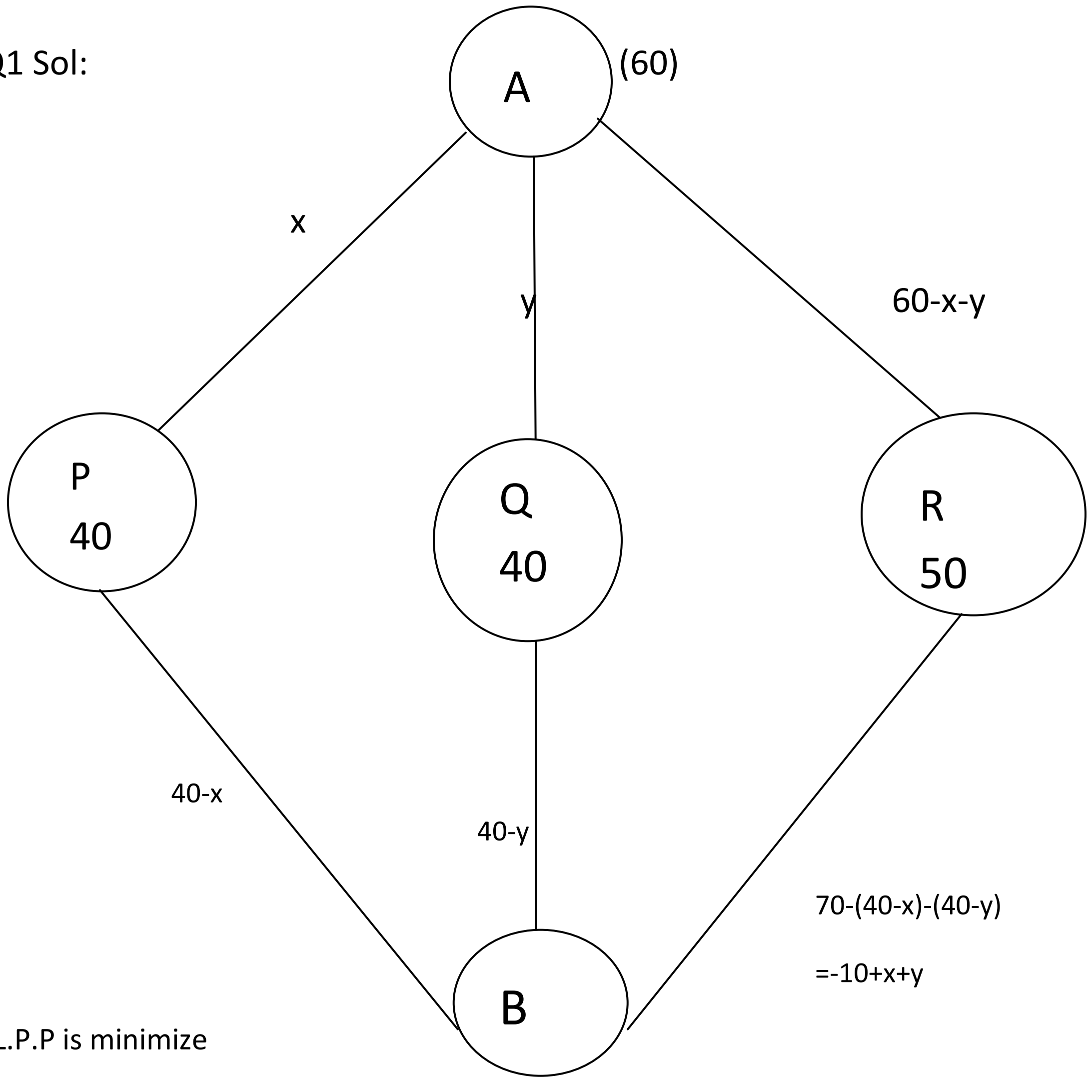
$\frac{375}{2} = 188\ EconomyclassPassengers$



TRANSPORTATION

LEVEL III

Q1 Sol:



The given L.P.P is minimize

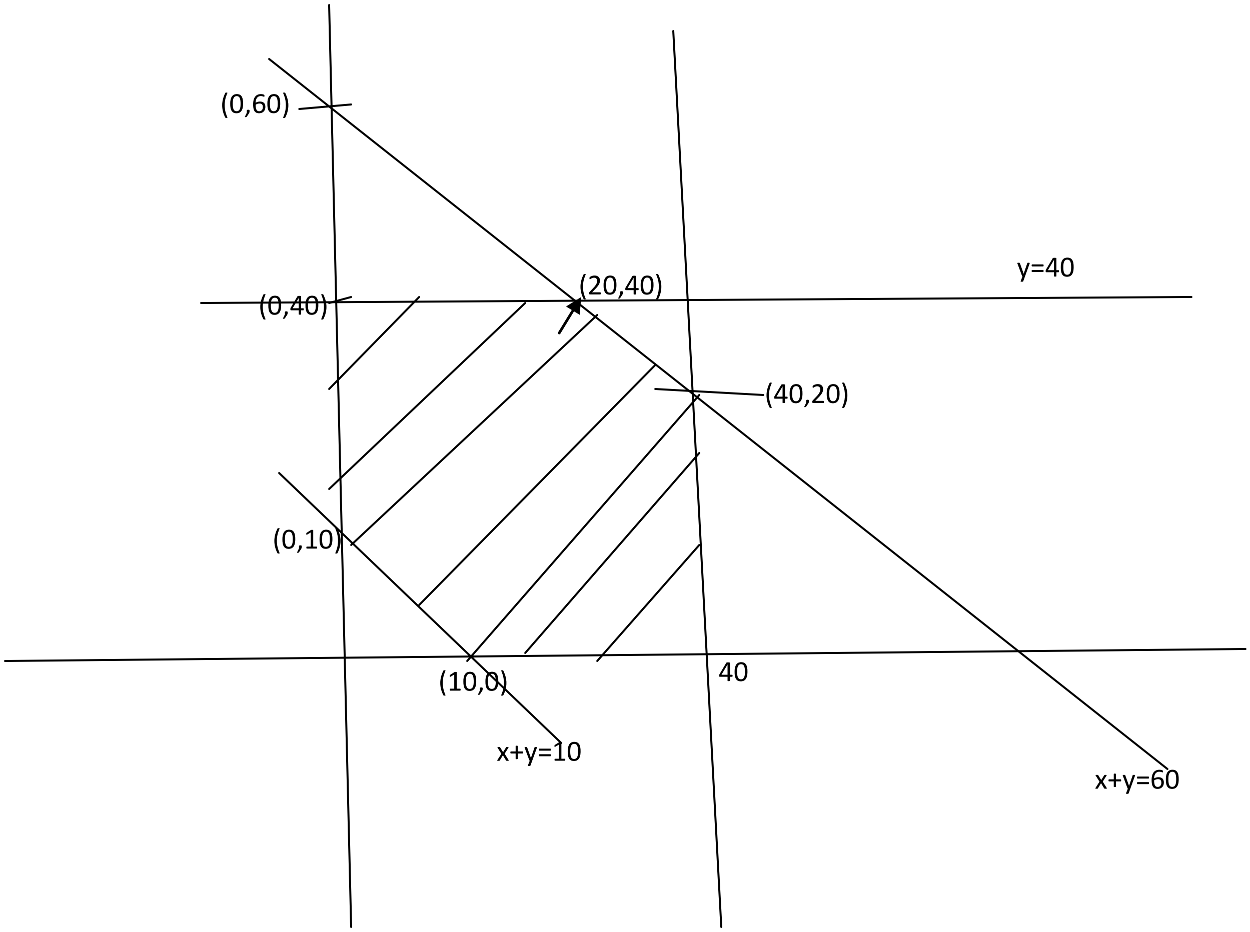
$$z = 5x + 4(40 - x) + 4y + 2(40 - y) + 3(60 - x - y) + 5(-10 + x + y)$$

$$= 3x + 4y + 370$$

Subject to $x + y \leq 60$

$$x \leq 40, y \leq 40$$

$$x + y \leq 10, x \geq 0, y \geq 0$$



Minimum transport cost =Rs 400

From A to P , Q & R . The no of packets transported 10,0,50 respectively

From B to P, Q &R , The no. of packets transported 30,40,0 respectively

TOPIC 12
PROBABILITY

(i) Conditional Probability

LEVEL I

Q1:P(A)=0.3,P(B)=.2

A & B ARE MUTUALLY EXCLUSIVE

$(A\cap B) = \emptyset$

$P(B/A) = \frac{P(A\cap B)}{P(A)} = 0$

Q2: Total no of balls =3R+5W

P(drawing 2 white balls)=$\frac{5C_2}{8\,C_2} = \frac{5}{14}$

LEVEL II

Q1:S={ (1,5),(2,4),(3,3),(4,2),(5,1)}

Favorable events={(2,4),(4,2)}

Reqd. Probality=$\frac{2}{5}$

LEVEL III

Q(1) P(A)= $\frac{5}{8}$, P(B)= $\frac{1}{2}$ P(A ∩B)= $1/2$

$P(\overline{A}/\bar{B}) = \frac{P(\bar{A}\cap\bar{B})}{P(\bar{B})} = \frac{P(AUB)^c}{1-P(B)} = \frac{1-P(AUB)}{1-P(B)}$=

=$\frac{3}{4}$

$P(\frac{\bar{B}}{\bar{A}}) = \frac{P(\bar{A}\cap\overline{B})}{P(\overline{A})} = \frac{\frac{3}{8}}{\frac{3}{8}} = 1$

(ii)Multiplication theorem on probability

LEVEL II

Q1: Total no of balls =5W+7R+3B

P(None of balls is red)=$\frac{8C_3}{15C_3}=\frac{8.7.6}{15.14.13}=\frac{8}{65}$

Q2: P(A)=$\frac{3}{7}$,P(B)=$\frac{1}{3}$

P($\overline{A}$) =$1-\frac{3}{7}=\frac{4}{7,}$ P($\overline{B}$)=$\frac{2}{3}$

P (Atleast one of them will hit the target)

= 1-P(none)=$1-\frac{4}{7}X\frac{2}{3}=1-\frac{8}{21}=\frac{13}{21}$

P(only one hit the target)=P(A).P($\bar{B}$) +P($\overline{A}$) P(B)

=$\frac{3}{7}.\frac{2}{3}+\frac{4}{7}.\frac{1}{3}$

$$\frac{6}{21}+\frac{4}{21}=\frac{10}{21}$$

LEVEL III

Q1:Consider the following events:

A= Selecting a fair complexioned student;

B =Selecting a rich student;

C =Selecting a girl.

We have P (A) =$\frac{1}{4}$ P (B) =$\frac{1}{8}$ and P (C) =$\frac{5}{16}$

Since A ,B , C are independent events. Therefore ,

Required Probability =P(A) P (B) P (C) =$\frac{1}{4}\times\frac{1}{8}\times\frac{5}{16}$ =$\frac{5}{512}$

Q2: Given numbers are 1,2,3,4,5,6,7,8,9,10,11

Odd numbers=6

Even numbers=5

Sum is even if both nos are odd or both numbers are even

$5C_2$+$6C_2$=10+15=25

Both nos are odd can be selected in

$=6C_2=\frac{6.5}{2}=15$

Required Probability$=\frac{15}{25}=\frac{3}{5}$

(iii) Independent Events

LEVEL I

Q1:S={HHH,HHT,HTH,THH,TTT,HTT,THT,TTH}

E= {HHH,HHT,HTT,HTH}

F= {TTT,THT,HTT,HHT}

$E\cap F=${HHT,HTT}

$P(E)=\frac{4}{8}=\frac{1}{2}$, $P(F)=\frac{4}{8}=\frac{1}{2}$

$$\therefore \left[P(E\cap F)=P(E).P(F)=\frac{2}{8}=\frac{1}{4}\right]$$

E And F are independent events.

Q2:$P(A)=\frac{1}{4}$,$P(B)=$,$\frac{2}{3}$,$P(A\cup B)=\frac{3}{4}$

P (AUB)=P(A)+P(B)-P(A$\cap B$)

$$\frac{3}{4}=\frac{1}{4}+\frac{2}{3}-P(A\cap B)\Longrightarrow P(A\cap B)=\frac{3+8-9}{12}=\frac{1}{6}$$

$NOW P(A).P(B)=\frac{1}{4}.\frac{2}{3}=\frac{1}{6}$YES, A & B are independent

$Q3: P(A)=0.35$

P(AUB)=0.60

$P(A\cap B)=P(A).P(B)$

Since A & B *are Independent events*.

P(AUB)=P(A)+P(B)-P(A$\cap B$)

.60=.35+P(B)-0.35XP(B)

0.60=.35+P(B)(1-0.35)

0.60-.35=P(B)(.65)

$$\frac{0.25}{0.65} = P(B). \Longrightarrow \frac{5}{13} = P(B).$$

(iv) Baye's theorem, partition of sample space and Theorem of total probability

LEVEL I

Q:1 Bag 1 Bag 2

Total no of Ball 6R+5B 5R+8B

Case 1 Red Ball transferred = bag 2 has 6R+8B

Case 2 Blue Ball transferred = bag 2 has 5R+9B

Case 1 P (Blue ball from bag 2) =$\frac{8C_1}{14C_1}*\frac{6C_1}{11C_1} = \frac{48}{154}$

Case 2 P (Blue ball from bag 2) =$\frac{9C_1}{14C_1}*\frac{5C_1}{11C_1} = \frac{45}{154}$

Required Probability=$\frac{48}{154} + \frac{45}{154} = \frac{93}{154}$

$Q2$: *let* E_1, E_2, E_3, E_4 *are the events that lost card is a diamond , heart, spade , club respectively*

Let A be the event of getting two Hearts Cards from the remaining 51 cards

=P(E_1),= $P(E_2)=P(E_3)=P(E_4)=\frac{13}{52}=\frac{1}{4}$

And p (A/E_1) = $\frac{13C_2}{51C_{2,}}$,P(A/E_2)=$\frac{12C_2}{51C_{2,}}$

P(A/E_3) = $\frac{13C_2}{51C_{2,}}$ AND P(A/E_4)=$\frac{13C_2}{51C_{2,}}$

Required Probability =P(E_2/A)

=$\frac{P(E_2).P(A/E_2)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+P(E_3).P\left(\frac{A}{E_3}\right)+P(E_4).P\left(\frac{A}{E_4}\right)}$=$\frac{11}{50}$

Q3 : let E_1, E_2, be the events of selection of a Scooter and a motorcycle

Let A =event that the insured vehicle met an accident.

P(E_1)=$\frac{2000}{5000} = \frac{2}{5}$

P(E_2)=$\frac{3000}{5000} = \frac{3}{5}$

P(A/E_1)=.01,P(A/E_2)=.02

∴Required Probablity =P(E_2/A)

=$\frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1)+P(E_2)P(A/E_2)}$

=$\frac{3}{4}$

Q4: Purse 1 purse 2

2S+4C 4S+3C

let E_1 , &E_2, be the events of selection of aPurse 1, Purse 2 respectively

P (E_1) = P ($E_2 = \frac{1}{2}$

∴ Probality(of getting a silver coin from purse1) + P(of getting a silver coin from purse2

=$\frac{2}{6} X \frac{1}{2} + \frac{4}{7} X \frac{1}{2}$

=$\frac{19}{42}$

Q5: Boys =$\frac{2}{3}$ Girls=$\frac{1}{3}$

P(of a girl getting first class)=0.25

P(of a boy getting first class)=0.28

∴Required Probablity==$\frac{81}{300} = \frac{2}{3} X0.28 + \frac{1}{3} X0.25 = \frac{81}{300} = .27\%$

LEVEL II

Q1: I compartment II compartment

3-fifty paisa coins 2- fifty paisa coins

+2 one rupee coin 3 one rupee coins

∴Required Probablity=$\frac{2}{5} X \frac{1}{2} + \frac{3}{5} X \frac{1}{2}$=$\frac{1}{2}$

Q2: let E_{1}, & E_{2}, be the events of selection of a Men & *women Orater*

P (E_1) =$\frac{1}{2}$

P (E_2) =$\frac{1}{2}$

Probability of good orator in men =$\frac{5}{100}$

Probability of good orator in women =$\frac{25}{1000}$

P(that male orator is selected) =$\frac{\frac{1}{2}*\frac{5}{100}}{\frac{1}{2}*\frac{5}{100}+\frac{1}{2}*\frac{25}{1000}}$=$\frac{2}{3}$

Q3: Plant 1 Plant 11

60% 40%

S=80% S=90%

∴Required Probablity OF Selected Cycle of standard Quality from Second pant

$$= \frac{\frac{40}{100} * \frac{90}{100}}{\frac{40}{100} * \frac{90}{100} + \frac{60}{100} * \frac{80}{100}}$$

LEVEL III

Q1: London Clifton

Favourable outcome=$\frac{2}{5}$ Favourable outcome $= \frac{1}{6}$

(i) ∴Required Probability it comes from

London =$\frac{\frac{2}{5}*\frac{1}{2}}{\frac{2}{5}*\frac{1}{2}+\frac{1}{6}*\frac{1}{2}} = \frac{12}{17}$

(ii) ∴Required Probability it comes from

Clifton =$\frac{\frac{1}{6}*\frac{1}{2}}{\frac{2}{5}*\frac{1}{2}+\frac{1}{6}*\frac{1}{2}} = \frac{5}{17}$

Q2 let $E_1 =$ the person has disease

$E_2 =$ the person selected does not have disease

A= test is positive

We have

$p(E_1) = 0.002, p(E_2) = 0.998$p (A/$E_1$) =0.90 ,P(A/E_2)= 0.01

Required Probability =P (E_1/A

=$\frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2)P(A/E_2)}$

=$\frac{0.002*0.90}{0.002*0.90+0.998*0.01}$

=$\frac{18}{10000*}*\frac{10000}{1178}$

= 0.15

Q3: Consider the following events :

E_1=Box I choosen

E_2=Box II choosen

E_3=Box III choosen

A=The coin drawn is of gold

$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(A/E_1)=1$ $P(A/E_2)=0$ $P(A/E_3)=\frac{1}{2}$

$P(E_1/A) = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$

(v) Random variables & probability distribution Mean & variance of random variables

LEVEL I

Q1 p= probability of getting a spade card

$=\frac{13}{52} = \frac{1}{4}$

$q=1-p=\frac{3}{4}$

X	0	1	2
P(x)	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

Q2:p= apple is defective $=\frac{4}{20} = \frac{1}{5}$

$q= 1 - \frac{1}{5} = \frac{4}{5}$

X	0	1	2	3
P(x)	$\frac{28}{57}$	$\frac{24}{57}$	$\frac{24}{285}$	$\frac{1}{285}$

Q3 $Var(x) = ?$

X	$P(x_i)$	$x_i p_i$	$x_i^2 p_i$
2	.3	.6	1.2
3	.4	1.2	3.6
4	.3	1.2	4.8
Total		3.0	9.6

Var(x)=$\sum(x_i)^2 p_i - [\sum(x_i p_i)^2)] = 9.6 - (3.0)^2 = 0.6$

Level III

Q1Sol: P (H) = $\frac{3}{4}$ P (T) =$\frac{1}{4}$

P = $\frac{1}{4}$q =$1 - \frac{1}{4} = \frac{3}{4}$

X	O	1	2
P(X)	$\frac{9}{16}\frac{6}{16}\frac{1}{16}$		

Q2 n=5

np+npq=1.8

5p+5p(1-p)=1.8

5p+5p-5p^2=1.8

10p-5p^2 =1.8

5p^2 -10p+1.8=0

p= $\frac{10\pm\sqrt{100-36}}{10}$

$$= \frac{(10 \pm 8)}{10} = \frac{18}{10}, \frac{2}{10}$$

=P=$\frac{18}{10}$ not possible

$$\therefore p = \frac{2}{10} = \frac{1}{5}$$

q=1-p=$1 - \frac{1}{5} = \frac{4}{5}$

∴Probabality distribution

=$(\frac{4}{5} + \frac{1}{5})^5$

Q3np =$\frac{4}{3}$, npq=$\frac{8}{9}$

q= $\frac{npq}{np} = \frac{\frac{8}{9}}{\frac{4}{3}} = \frac{2}{3}$

P= $1-\frac{2}{3}=\frac{1}{3}$

n=4

$$P(x \geq 1) \ = \ 1-p(x=0)$$

= 1-4$c_0 q^4 p^0$

=1-$\left(\frac{2}{3}\right)^4$

= $1-\frac{16}{81}=\frac{81-16}{81}=\frac{65}{81}$

(vi) Bernoulli,s trials and Binomial Distribution

LEVEL II

Q1:p=probability getting even no=$\frac{1}{2}$

P (getting even no exactly 3 times)

$$5C_3 \ q^{5-3}p^3$$

= $\frac{\angle 5}{\angle 3 \angle 2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3=\frac{10}{32}=\frac{5}{16}$

Q2: p= prob of success =$\frac{2}{3}.q=1-p=\frac{1}{3}$

P(at least 4 successes)=p(x=4)+p(x=5)+p(x=6)

=$6c_4 q^2 p^4+6c_5 q p^5+6c_6 p^6$

=$\frac{6.5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^6+6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5$

$=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$+6.$\frac{1}{3}$.$\left(\frac{2}{3}\right)^5$+$\left(\frac{2}{3}\right)^6$

=$\frac{15.16}{3^6}$+$\frac{6.32}{3^6}+\frac{64}{3^6}=\frac{240+192+64}{3^6}$

=$\frac{496}{729}$

Q3: Sum 9 will be in(3,6),(4,5),(5,4),(6,3)

p= getting sum as 9=$\frac{4}{36}=\frac{1}{9}$

$q=1-\frac{1}{9}=\frac{8}{9}$,n=200

Mean= np , Variance=npq

$= 200*\frac{1}{9}$ $= 200*\frac{1}{9}*\frac{8}{9}$

$=\frac{200}{9}$, $\frac{1600}{81}$